Problem Set

Chapter 04.05
System of Equations

1. For a set of equations $[A][X]=[B]$, a unique solution exists if
   (A) rank (A) = rank $(A \vdots B)$
   (B) rank (A) = rank $(A \vdots B)$ and rank (A) = number of unknowns
   (C) rank (A) = rank $(A \vdots B)$ and rank (A) = number of rows of (A).

2. The rank of matrix
   $$A=\begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix} \text{ is}$$
   (A) 1
   (B) 2
   (C) 3
   (D) 4

3. A 3×4 matrix can have a rank of at most
   (A) 3
   (B) 4
   (C) 5
   (D) 12

4. If $[A][X]=[C]$ has a unique solution, where the order of $[A]$ is 3×3, $[X]$ is 3×1, then the rank of $[A]$ is
   (A) 2
   (B) 3
   (C) 4
   (D) 5

5. Show if the following system of equations is consistent or inconsistent. If they are consistent, determine if the solution would be unique or infinite ones exist.
   $$\begin{bmatrix} 1 & 2 & 5 \\ 7 & 3 & 9 \\ 8 & 5 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 8 \\ 19 \\ 27 \end{bmatrix}$$

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8. The set of equations

$$\begin{bmatrix} 1 & 2 & 5 \\ 7 & 3 & 9 \\ 8 & 5 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 19 \\ 27 \end{bmatrix}$$

has
(A) Unique solution
(B) No solution
(C) Infinite solutions

9. For what values of a will the following equation have

$$x_1 + x_2 + x_3 = 4$$
$$x_3 = 2$$
$$\left(a^2 - 4\right)x_1 + x_3 = a - 2$$

(A) Unique solution
(B) No solution
(C) Infinite solutions

10. Find if

$$[A] = \begin{bmatrix} 5 & -2.5 \\ -2 & 3 \end{bmatrix}$$

and

$$[B] = \begin{bmatrix} 0.3 & 0.25 \\ 0.2 & 0.5 \end{bmatrix}$$

are inverse of each other.

11. Find if

$$[A] = \begin{bmatrix} 5 & 2.5 \\ 2 & 3 \end{bmatrix}$$

and

$$[B]=\begin{bmatrix}0.3 & -0.25\\ 0.2 & 0.5\end{bmatrix}$$

are inverse of each other.

12. Find the
    (A) cofactor matrix
    (B) adjoint matrix

    of

$$[A]=\begin{bmatrix}3 & 4 & 1\\ 2 & -7 & -1\\ 8 & 1 & 5\end{bmatrix}$$

13. Find $[A]^{-1}$ using any method for

$$[\mathrm{A}]=\begin{bmatrix}3 & 4 & 1\\ 2 & -7 & -1\\ 8 & 1 & 5\end{bmatrix}$$

14. Prove that if $[A]$ and $[B]$ are both invertible and are square matrices of same order, then

$$([A][B])^{-1}=[B]^{-1}[A]^{-1}$$

15. What is the inverse of a square diagonal matrix? Does it always exist?

16. $[A]$ and $[B]$ are square matrices. If $[A][B]=[0]$ and $[A]$ is invertible, show $[B]=0$.

17. If $[A][B][C]=[I]$, where $[A]$, $[B]$ and $[C]$ are of the same size, show that $[B]$ is invertible.

18. Prove if $[B]$ is invertible, $[A][B]^{-1}=[B]^{-1}[A]$ if and only if $[A][B]=[B][A]$

19. For

$$[A]=\begin{bmatrix}10 & -7 & 0\\ -3 & 2.099 & 6\\ 5 & -1 & 5\end{bmatrix}$$

$$[A]^{-1}=\begin{bmatrix}-0.1099 & -0.2333 & 0.2799\\ -0.2999 & -0.3332 & 0.3999\\ 0.04995 & 0.1666 & 6.664\times10^{-5}\end{bmatrix}$$

Show

$$\det(A) = \frac{1}{\det(A^{-1})}.$$

20. For what values of a does the linear system have
$x + y = 2$
$6x + 6y = a$
(A) infinite solutions
(B) unique solution

21. Three kids - Jim, Corey and David receive an inheritance of $2,253,453. The money is put in three trusts but is not divided equally to begin with. Corey gets three times more than David because Corey made an "A" in Dr. Kaw's class. Each trust is put in an interest generating investment. The three trusts of Jim, Corey and David pays an interest of 6%, 8%, 11%, respectively. The total interest of all the three trusts combined at the end of the first year is $190,740.57. How much money was invested in each trust? Set the following as equations in a matrix form. Identify the unknowns. Do not solve for the unknowns.

22. What is the rank of
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 6 & 10 & 13 \end{bmatrix}?$$
Justify your answer.

23. What is the rank of
$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 4 & 6 & 7 & 17 \\ 6 & 10 & 13 & 29 \end{bmatrix}?$$
Justify your answer.

24. What is the rank of
$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 4 & 6 & 7 & 18 \\ 6 & 10 & 13 & 30 \end{bmatrix}?$$
Justify your answer.

25. How many solutions does the following system of equations have
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 6 & 10 & 13 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \\ 29 \end{bmatrix}?$$

Justify your answer.

26. How many solutions does the following system of equations have
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 6 & 10 & 13 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \\ 30 \end{bmatrix}?$$
Justify your answer.

27. By any scientific method, find the second column of the inverse of
$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 13 \end{bmatrix}.$$

28. Just write out the inverse of (no need to show any work)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

29. Solve $[A][X]=[B]$ for $[X]$ if
$$[A]^{-1} = \begin{bmatrix} 10 & -7 & 0 \\ 2 & 2 & 5 \\ 2 & 0 & 6 \end{bmatrix}$$
and
$$[B] = \begin{bmatrix} 7 \\ 2.5 \\ 6.012 \end{bmatrix}$$

30. Let $[A]$ be a 3×3 matrix. Suppose
$$[X] = \begin{bmatrix} 7 \\ 2.5 \\ 6.012 \end{bmatrix}$$
is a solution to the homogeneous set of equations $[A][X]=[0]$ (the right hand side is a zero vector of order 3×1). Does $[A]$ have an inverse?
Justify your answer.

31. Is the set of vectors

$$\vec{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{B} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \vec{C} = \begin{bmatrix} 1 \\ 4 \\ 25 \end{bmatrix}$$

linearly independent? Justify your answer.

32. What is the rank of the set of vectors

$$\vec{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{B} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \vec{C} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}?$$

Justify your answer.

33. What is the rank of

$$\vec{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{B} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \vec{C} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix}?$$

Justify your answer.

Answers to Selected Problems:

1. B
2. A
3. A
4. B
5. Consistent; Infinite solutions
6. Inconsistent
7. Consistent; Unique
8. C
9. If $a \neq +2\ or -2,$ then there will be a unique solution
   If $a = +2$ or - 2, then there will be no solution.
   Possibility of infinite solutions does not exist.
10. Yes
11. No
12. $\begin{bmatrix} -34 & -18 & 58 \\ -19 & 7 & 29 \\ 3 & 5 & -29 \end{bmatrix} \begin{bmatrix} -34 & -19 & 3 \\ -18 & 7 & 5 \\ 58 & 29 & -29 \end{bmatrix}$
13. $[\mathrm{A}]^{-1} = \begin{bmatrix} 2.931\times10^{-1} & 1.638\times10^{-1} & -2.586\times10^{-2} \\ 1.552\times10^{-1} & -6.034\times10^{-2} & -4.310\times10^{-2} \\ -5.000\times10^{-1} & -2.500\times10^{-1} & 2.500\times10^{-1} \end{bmatrix}$
14. $([A][B])^{-1} = [B]^{-1}[A]^{-1}$

    Let $[C] = [A][B]$

    $[C][B]^{-1} = [A][B][B]^{-1}$

    $= [A][I]$

    $= [A]$

    Again

    $[C] = [A][B]$

$$[A]^{-1}[C] = [A]^{-1}[A][B]$$
$$= [I][B]$$
$$= [B]$$

So

$$[C][B]^{-1} = [A] \qquad (1)$$
$$[A]^{-1}[C] = [B] \qquad (2)$$

From (1) and (2)

$$[C][B]^{-1}[A]^{-1}[C] = [A][B]$$
$$[A][B][B]^{-1}[A]^{-1}[A][B] = [A][B]$$
$$[A]^{-1}[A][B][B]^{-1}[A]^{-1}[A][B] = [A^{-1}][A][B]$$
$$[B][B]^{-1}[A^{-1}][A][B] = [B]$$
$$[B^{-1}][B][B]^{-1}[A^{-1}][A][B] = [B]^{-1}[B]$$
$$[B]^{-1}[A^{-1}][A][B] = [I].$$

15. **Hint:** Inverse of a square $n \times n$ diagonal matrix $[A]$ is $[A]^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & \cdots & \cdots & \frac{1}{a_{nn}} \end{bmatrix}$

So inverse exists only if $a_{ii} \neq 0$ for all i.

16. $$[A][B] = [0]$$
$$[A^{-1}][A][B] = [A]^{-1}[0]$$

17. Hint: $\det(AB) = \det(A)\det(B)$

18. Hint: Multiply by $[B]^{-1}$ on both sides, $[A][B][B]^{-1} = [B]^{-1}[A][B]^{-1}$

19.

20.
(A) 12
(B) not possible

21. J + C + D = $2,253,453
C = 3D
0.06J+0.08C+0.11D = $190,740.57
In matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0.06 & 0.08 & 0.11 \end{bmatrix} \begin{bmatrix} J \\ C \\ D \end{bmatrix} = \begin{bmatrix} 2{,}253{,}453 \\ 0 \\ 190{,}740.57 \end{bmatrix}$$

22. In the above matrix, 2(Row 1) + Row 2 = Row 3. Hence, rank is less than 3. Row 1 and Row 2 are linearly independent. Hence, the rank of the matrix is 2.

23. The determinant of all the 3×3 sub-matrices is zero. Hence, the rank is less than 3. Determinant of
$$\begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} = -4 \neq 0 .$$
Hence, the rank is 2.

24. In the above matrix, 2(Row 1) + Row 2 = Row 3. Hence, rank is less than 3 as the 3 rows are linearly dependant. Determinant of
$$\begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} = -4 \neq 0 .$$
Hence, the rank is 2.

25. Rank of A = 2
Rank of A|C = 2
Number of unknowns = 3.
There are infinite solutions since rank of A is less than the number of unknowns.

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27.
$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 13 \end{bmatrix} \begin{bmatrix} X & a'_{12} & X \\ X & a'_{22} & X \\ X & a'_{32} & X \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$a'_{12} + 2a'_{22} = 0$$
$$4a'_{12} + 5a'_{22} = 1$$
$$13a'_{32} = 0$$

Simplifying,

$$\begin{bmatrix} a'_{12} \\ a'_{22} \\ a'_{32} \end{bmatrix} = \begin{bmatrix} 0.667 \\ -0.333 \\ 0 \end{bmatrix}$$

28. $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{5} \end{bmatrix}$$

29. $$[X] = [A]-1[B]$$
$$= \begin{bmatrix} 10 & -7 & 0 \\ 2 & 2 & 5 \\ 2 & 0 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 2.5 \\ 6.012 \end{bmatrix}$$
$$= \begin{bmatrix} 52.5 \\ 49.06 \\ 50.072 \end{bmatrix}$$

30. Given
$$[A][X] = [0]$$
If $[A]^{-1}$ exists, then
$$[A]^{-1}[A][X] = [A]^{-1}[0]$$
$$[I][X] = [0]$$
$$[X] = [0]$$
This contradicts the given value of $[X]$. Hence, $[A]^{-1}$ does not exist.

31. The set of vectors are linearly independent.

32. Since, the 3 vectors are linearly independent as proved above, the rank of the 3 vectors is 3.

33. By inspection, $\vec{C} = \vec{A} + \vec{B}$. Hence, the 3 vectors are linearly dependent, and the rank is less than 3. Linear combination of $\vec{A}$ and $\vec{B}$, that is, $K_1\vec{A} + K_2\vec{B} = 0$ has only one solution $K_1 = K_2 = 0$. Therefore, the rank is 2.