Problem Set

Chapter 04.09
Adequacy of Solutions

1. The adequacy of the solution of simultaneous linear equations depends on
   (A) Condition number
   (B) Machine epsilon
   (C) Product of condition number and machine epsilon
   (D) Norm of the matrix.

2. If a system of equations $[A][X]=[C]$ is ill conditioned, then
   (A) $\det(A)=0$
   (B) $\text{Cond}(A)=1$
   (C) $\text{Cond}(A)$ is large.
   (D) $\|A\|$ is large.

3. If $\text{Cond}(A)=10^4$ and $\in_{mach}=0.119\times10^{-6}$, then in $[A][X]=[C]$, at least these many significant digits are correct in your solution,
   (A) 0
   (B) 1
   (C) 2
   (D) 3

4. Make a small change in the coefficient matrix of $\begin{bmatrix}1 & 2\\ 2 & 3.999\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4\\ 7.999\end{bmatrix}$
   and find
   $$\frac{\|\Delta X\|_\infty/\|X\|_\infty}{\|\Delta A\|_\infty/\|A\|_\infty}$$
   Is it a large or small number? How is this number related to the condition number of the coefficient matrix?

5. Make a small change in the coefficient matrix of
   $$\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4\\ 7\end{bmatrix}$$
   and find
   $$\frac{\|\Delta X\|_\infty/\|X\|_\infty}{\|\Delta A\|_\infty/\|A\|_\infty}.$$
   Is it a large or a small number? Compare your results with the previous problem. How is this number related to the condition number of the coefficient matrix?

6. Prove

$$\frac{\|\Delta X\|}{\|X\|} \leq \|A\| \|A^{-1}\| \frac{\|\Delta C\|}{\|C\|}$$

7. For

$$[A] = \begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix}$$

gives

$$[A]^{-1} = \begin{bmatrix} -0.1099 & -0.2333 & 0.2799 \\ -0.2999 & -0.3332 & 0.3999 \\ 0.04995 & 0.1666 & 6.664\times10^{-5} \end{bmatrix}$$

(A) What is the condition number of $[A]$?
(B) How many significant digits can we at least trust in the solution of $[A][X]=[C]$ if $\in_{mach} = 0.1192\times10^{-6}$?
(C) Without calculating the inverse of the matrix $[A]$, can you estimate the condition number of $[A]$ using the theorem in problem#6?

8. Prove that the $Cond(A) \geq 1$.

Answers to Selected Problems:

1. C
2. C
3. C
4. Changing $[A]$ to
$$\begin{bmatrix} 1.001 & 2.001 \\ 2.001 & 4.000 \end{bmatrix}$$
Results in solution of
$$\begin{bmatrix} 5999 \\ -2999 \end{bmatrix}$$
$$\frac{\|\Delta X\|_\infty / \|X\|_\infty}{\|\Delta A\|_\infty / \|A\|_\infty} = \frac{5999.7/2}{0.002/5.999}$$
$$= 8.994 \times 10^6$$

5. Changing $[A]$ to
$$\begin{bmatrix} 1.001 & 2.001 \\ 2.001 & 3.000 \end{bmatrix}$$
Results in solution of
$$\begin{bmatrix} 2.003 \\ 0.9970 \end{bmatrix}$$
$$\frac{\|\Delta X\|_\infty / \|X\|_\infty}{\|\Delta A\|_\infty / \|A\|_\infty} = \frac{0.003/2.000}{0.002/5}$$
$$= 3.75$$
6. Use theorem that if $[A][B] = [C]$ then $\|A\|\|B\| \geq \|C\|$
7.

(A) $\|A\| = 17$

$\|A^{-1}\| = 1.033$

Cond (A) = 17.56

(B) 5

(C) Try different values of right hand side of $C = [\pm 1 \ \pm 1 \ \pm 1]^{\mathrm{T}}$ with signs chosen randomly. Then $\|A^{-1}\| \leq \|X\|$ obtained from solving equation set $[A][X] = [C]$ as $\|C\| = 1$.

8. We know that

$$\|A\,B\| \leq \|A\|\,\|B\|$$

then if

$$[B] = [A]^{-1},$$

$$\|A\,A^{-1}\| \leq \|A\|\,\|A^{-1}\|$$

$$\|I\| \leq \|A\|\,\|A^{-1}\|$$

$$1 \leq \|A\|\,\|A^{-1}\|$$

$$\|A\|\,\|A^{-1}\| \geq 1$$

$$Cond(A) \geq 1$$