Newton-Raphson Method

Chemical Engineering Majors

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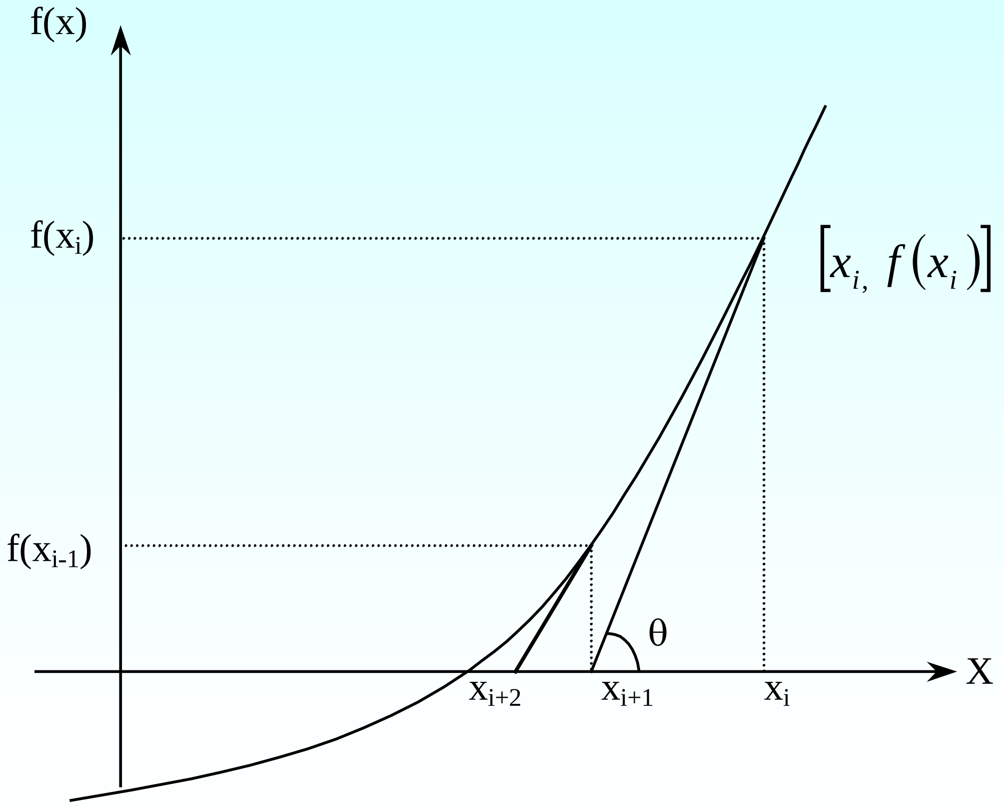
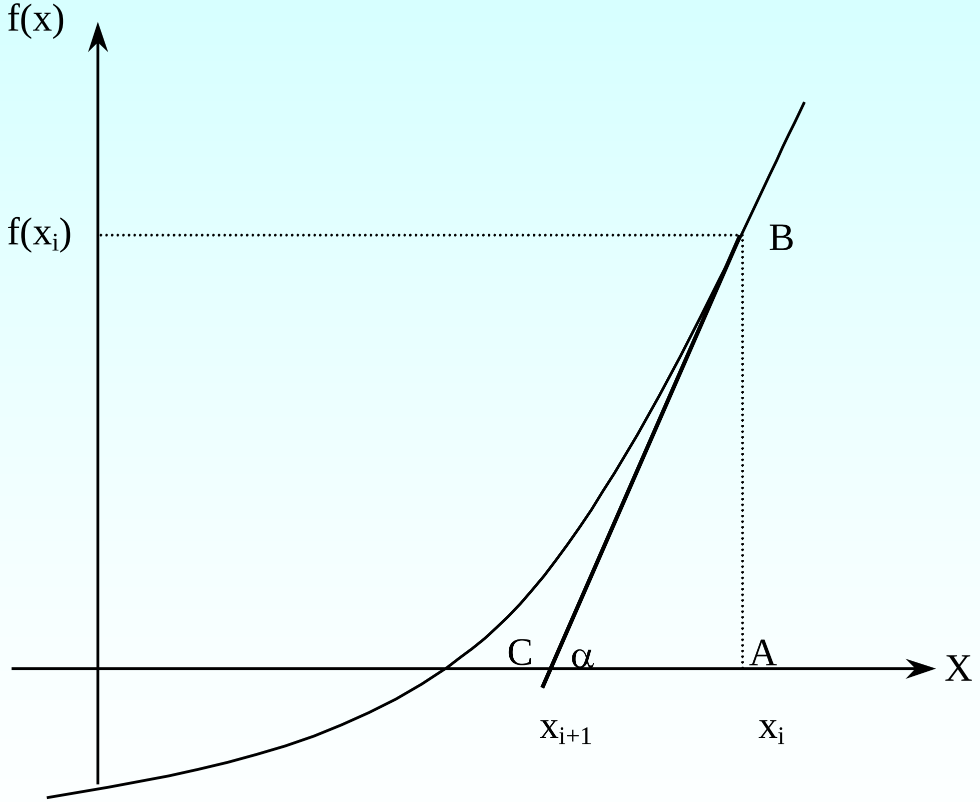


$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 1 Geometrical illustration of the Newton-Raphson method.

Derivation



$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 2 Derivation of the Newton-Raphson method.

Algorithm for Newton-Raphson Method

Step 1

Evaluate $f'(x)$ symbolically.

Step 2

Use an initial guess of the root, x_i , to estimate the new value of the root, x_{i+1} , as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

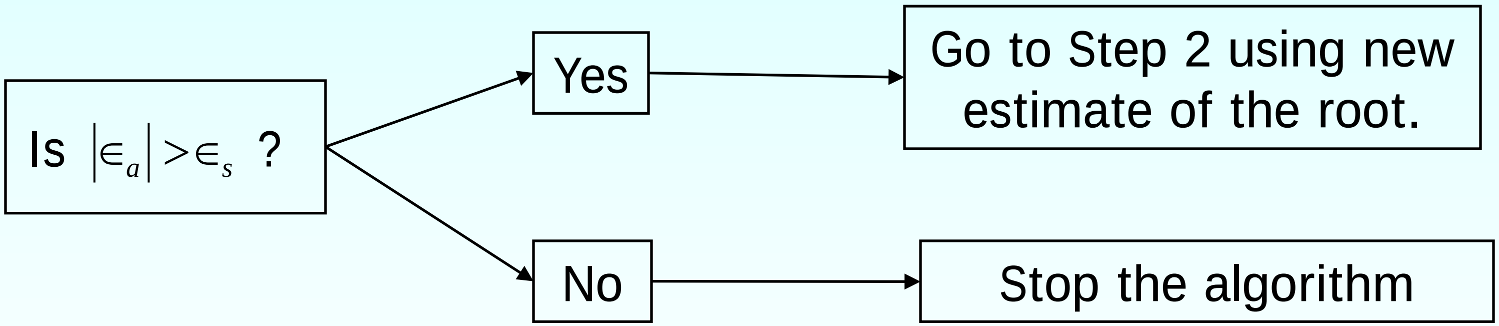
Step 3

Find the absolute relative approximate error $\left|\in_a\right|$ as

$$\left|\in_a\right| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$

Step 4

Compare the absolute relative approximate error with the pre-specified relative error tolerance $\in_s$.



Also, check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user.

Example 1

You have a spherical storage tank containing oil. The tank has a diameter of 6 ft. You are asked to calculate the height, h, to which a dipstick 8 ft long would be wet with oil when immersed in the tank when it contains 4 ft^3 of oil.

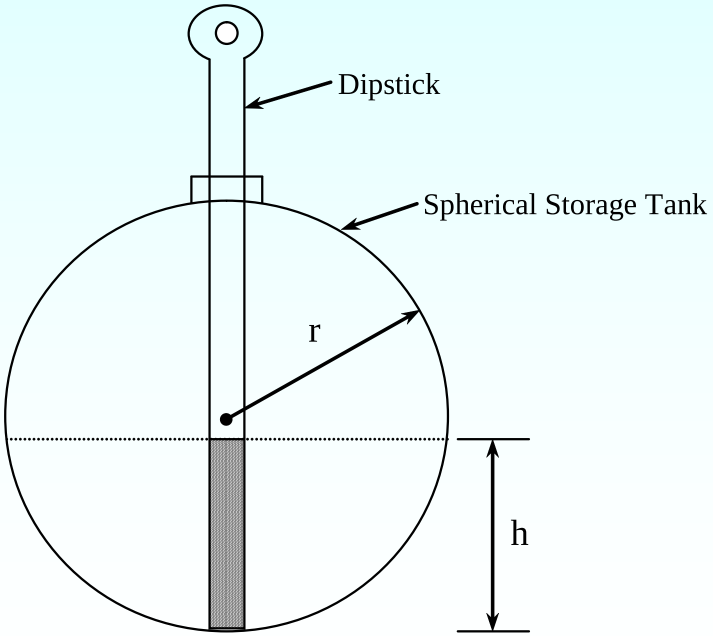


Figure 3 Spherical storage tank problem.

Example 1 Cont.

The equation that gives the height, *h*, of liquid in the spherical tank for the given volume and radius is given by

$$f(h) = h^3 - 9h^2 + 3.8197 = 0$$

Use the Newton-Raphson method of finding roots of equations to find the height, *h*, to which the dipstick is wet with oil. Conduct three iterations to estimate the root of the above equation. Find the absolute approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

Example 1 Cont.

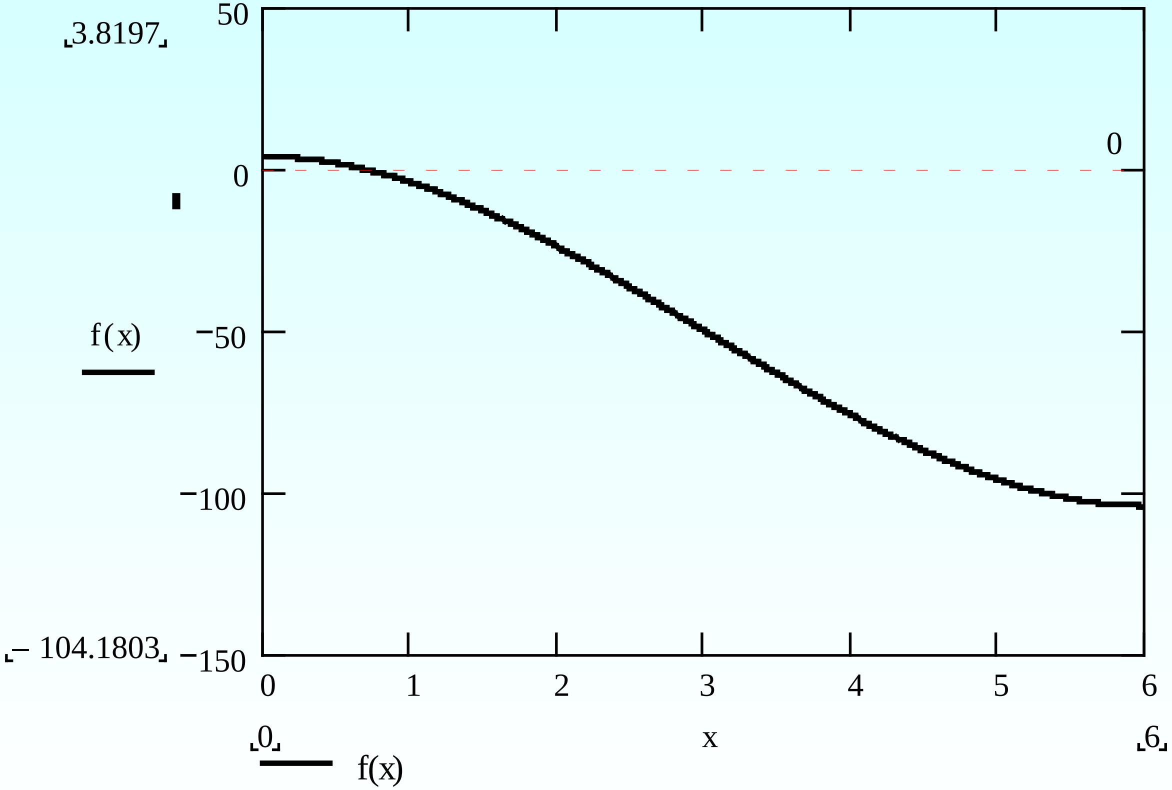


Figure 4 Graph of the function f(h).

$$f(h) = h^3 - 9h^2 + 3.8197 = 0$$

Example 1 Cont.

Solution

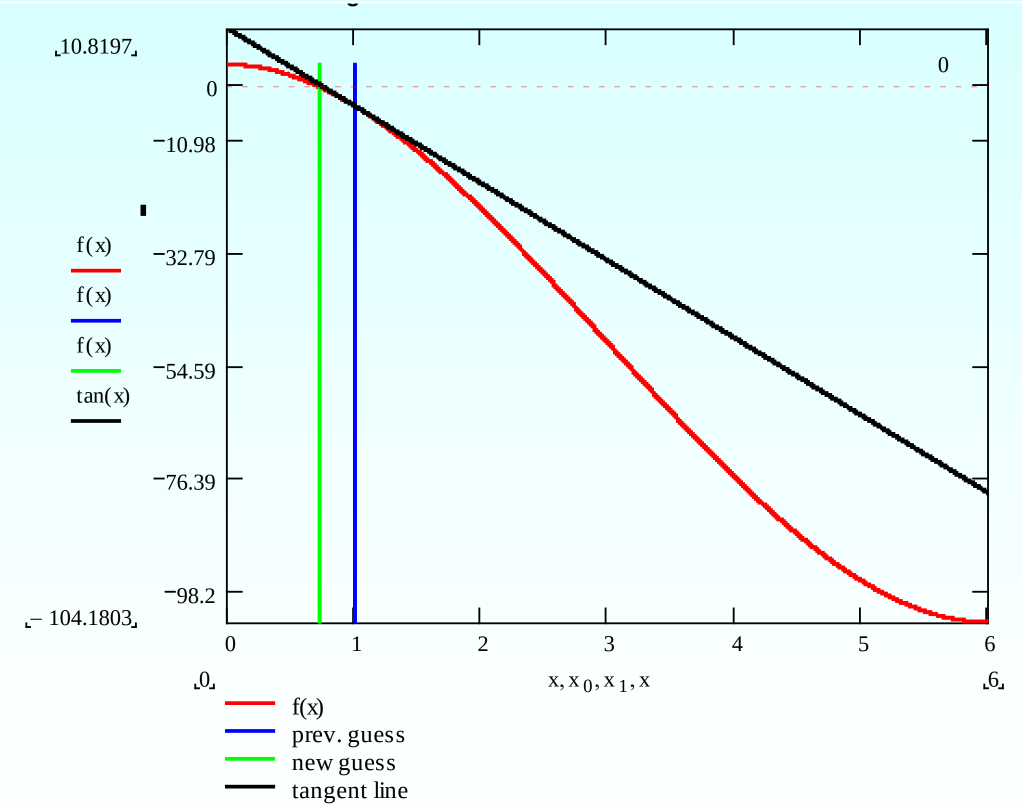


Figure 5 Graph of the estimated root after Iteration 1.

<u>Iteration 1</u>

The estimate of the root is

$$h_1 = h_0 - \frac{f(h_0)}{f'(h_0)}$$

$$= 1 - \frac{(h_0)^3 - 9(h_0)^2 + 3.8197}{3(h_0)^2 - 18(h_0)}$$

$$= 0.72131$$

The absolute relative approximate error is

$$|\in_a| = \left|\frac{h_1 - h_0}{h_1}\right| \times 100$$

$$= 38.636\,\%$$

The number of significant digits at least correct is 0.

Example 1 Cont.

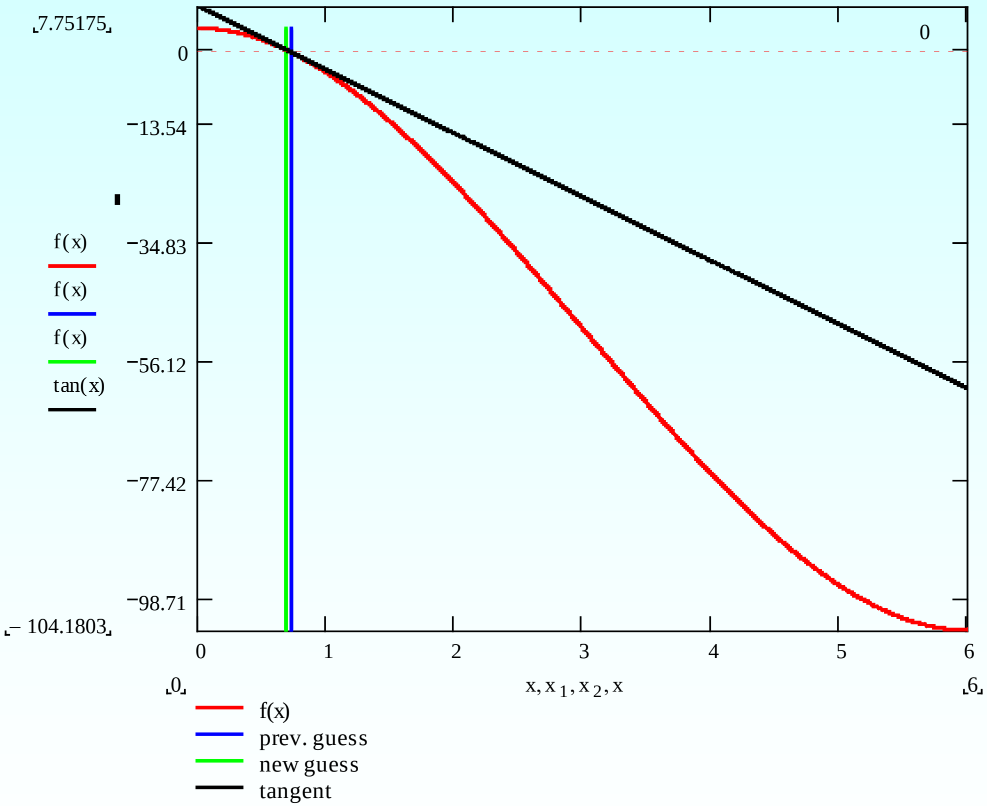


Figure 6 Graph of the estimated root after Iteration 2.

<u>Iteration 2</u>

The estimate of the root is

$$h_2 = h_1 - \frac{f(h_1)}{f'(h_1)}$$

$$= h_1 - \frac{(h_1)^3 - 9(h_1)^2 + 3.8197}{3(h_1)^2 - 18(h_1)}$$

$$= 0.67862$$

The absolute relative approximate error is

$$|\in_a| = \left|\frac{h_2 - h_1}{h_2}\right| \times 100$$

$$= 6.2907\%$$

The number of significant digits at least correct is 0.

Example 1 Cont.

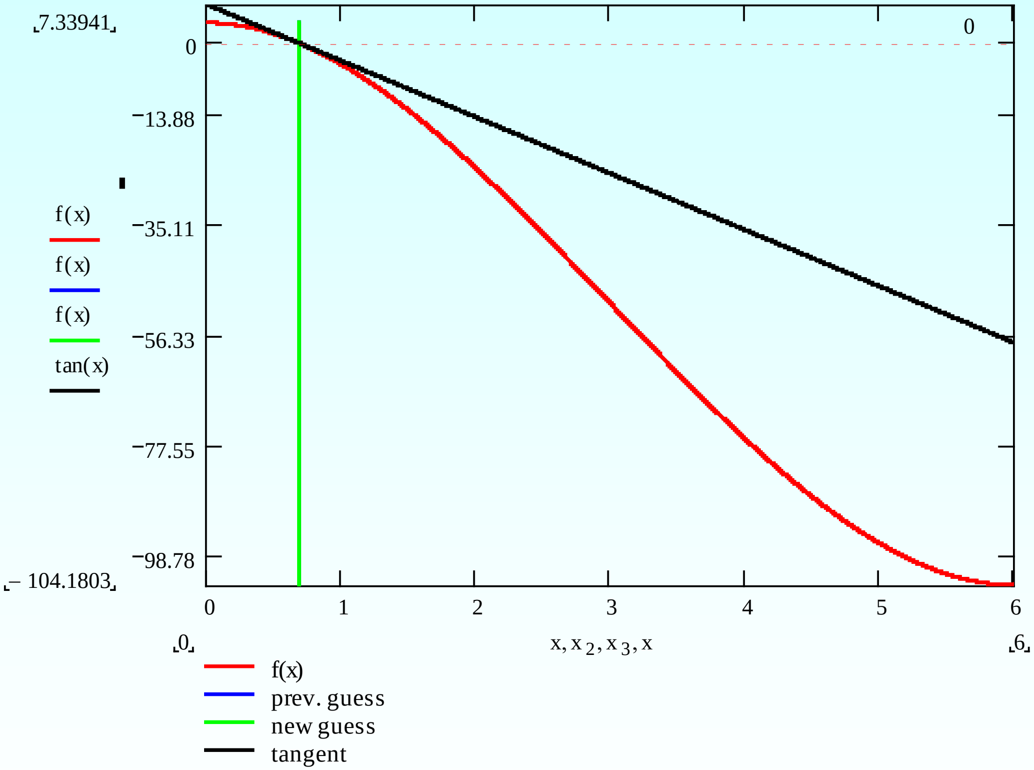


Figure 7 Graph of the estimated root after Iteration 3.

<u>Iteration 3</u>

The estimate of the root is

$$h_3 = h_2 - \frac{f(h_2)}{f'(h_2)}$$

$$= h_2 - \frac{(h_2)^3 - 9(h_2)^2 + 3.8197}{3(h_2)^2 - 18(h_2)}$$

$$= 0.67747$$

The absolute relative approximate error is

$$|\in_a| = \left|\frac{h_2 - h_1}{h_2}\right| \times 100$$

$$= 0.17081\%$$

The number of significant digits at least correct is 2.

Advantages and Drawbacks of Newton Raphson Method

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Advantages

- Converges fast (quadratic convergence), if it converges.
- Requires only one guess

Drawbacks

1. <u>Divergence at inflection points</u>
   Selection of the initial guess or an iteration value of the root that is close to the inflection point of the function $f(x)$ may start diverging away from the root in ther Newton-Raphson method.

For example, to find the root of the equation $f(x) = (x-1)^3 + 0.512 = 0$.

The Newton-Raphson method reduces to $x_{i+1} = x_i - \dfrac{(x_i^3 - 1)^3 + 0.512}{3(x_i - 1)^2}$.

Table 1 shows the iterated values of the root of the equation.

The root starts to diverge at Iteration 6 because the previous estimate of 0.92589 is close to the inflection point of $x = 1$.

Eventually after 12 more iterations the root converges to the exact value of $x = 0.2$.

Drawbacks – Inflection Points

Table 1 Divergence near inflection point.

Iteration Number	x_i
0	5.0000
1	3.6560
2	2.7465
3	2.1084
4	1.6000
5	0.92589
6	−30.119
7	−19.746
18	0.2000

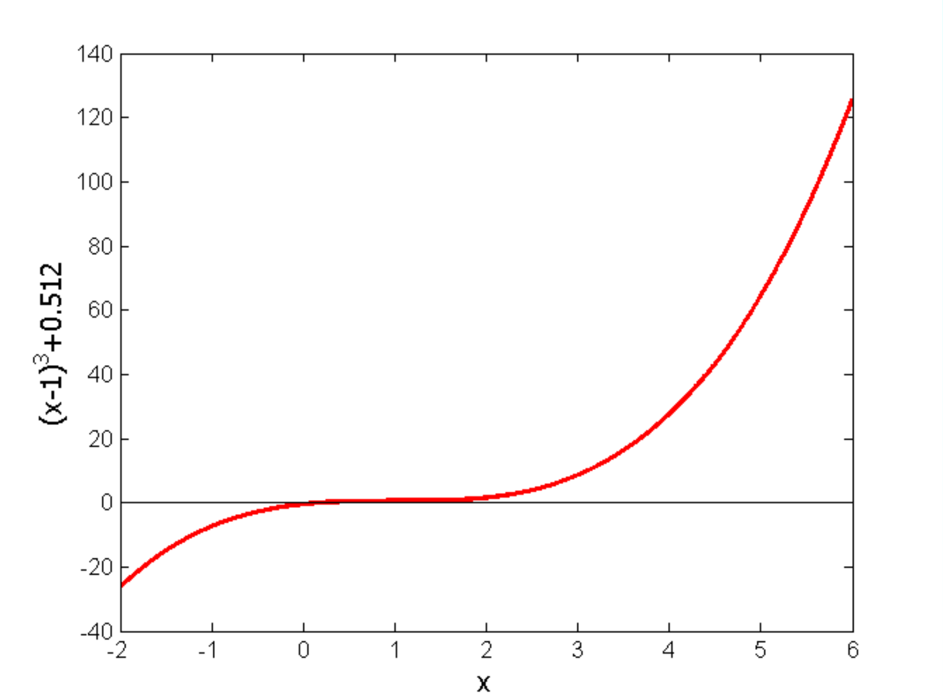

Figure 8 Divergence at inflection point for

$$f(x)=(x-1)^3+0.512=0$$

Drawbacks – Division by Zero

2. Division by zero
   For the equation

$$f(x) = x^3 - 0.03x^2 + 2.4\times10^{-6} = 0$$

the Newton-Raphson method reduces to

$$x_{i+1} = x_i - \frac{x_i^3 - 0.03x_i^2 + 2.4\times10^{-6}}{3x_i^2 - 0.06x_i}$$

For $x_0 = 0$ or $x_0 = 0.02$, the denominator will equal zero.

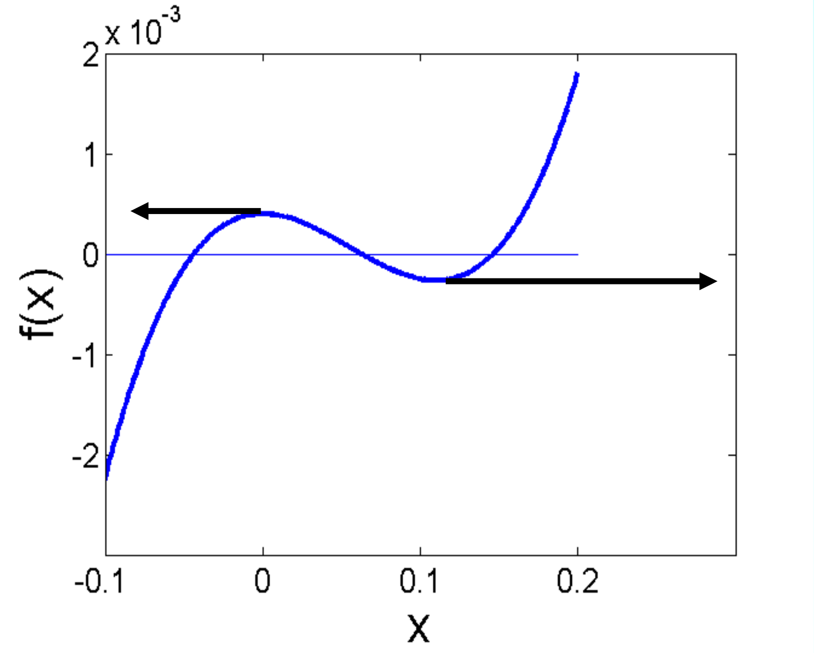

Figure 9 Pitfall of division by zero or near a zero number

Drawbacks – Oscillations near local maximum and minimum

3. <u>Oscillations near local maximum and minimum</u>

Results obtained from the Newton-Raphson method may oscillate about the local maximum or minimum without converging on a root but converging on the local maximum or minimum.

Eventually, it may lead to division by a number close to zero and may diverge.

For example for $f(x) = x^2 + 2 = 0$ the equation has no real roots.

Drawbacks – Oscillations near local maximum and minimum

Table 3 Oscillations near local maxima and mimima in Newton-Raphson method.

Iteration Number	x_i	$f(x_i)$	$\lvert \in_a \rvert \%$
0	–1.0000	3.00	
1	0.5	2.25	300.00
2	–1.75	5.063	128.571
3	–0.30357	2.092	476.47
4	3.1423	11.874	109.66
5	1.2529	3.570	150.80
6	–0.17166	2.029	829.88
7	5.7395	34.942	102.99
8	2.6955	9.266	112.93
9	0.97678	2.954	175.96

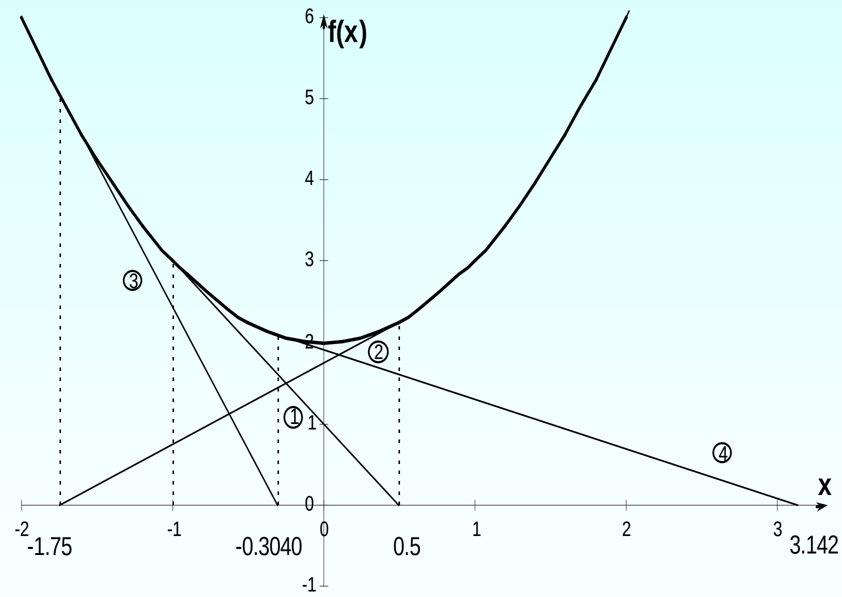

Figure 10 Oscillations around local minima for $f(x) = x^2 + 2$.

Drawbacks – Root Jumping

4. Root Jumping

In some cases where the function $f(x)$ is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root.

For example

$$f(x) = \sin x = 0$$

Choose

$$x_0 = 2.4\pi = 7.539822$$

It will converge to $x = 0$

instead of $x = 2\pi = 6.2831853$

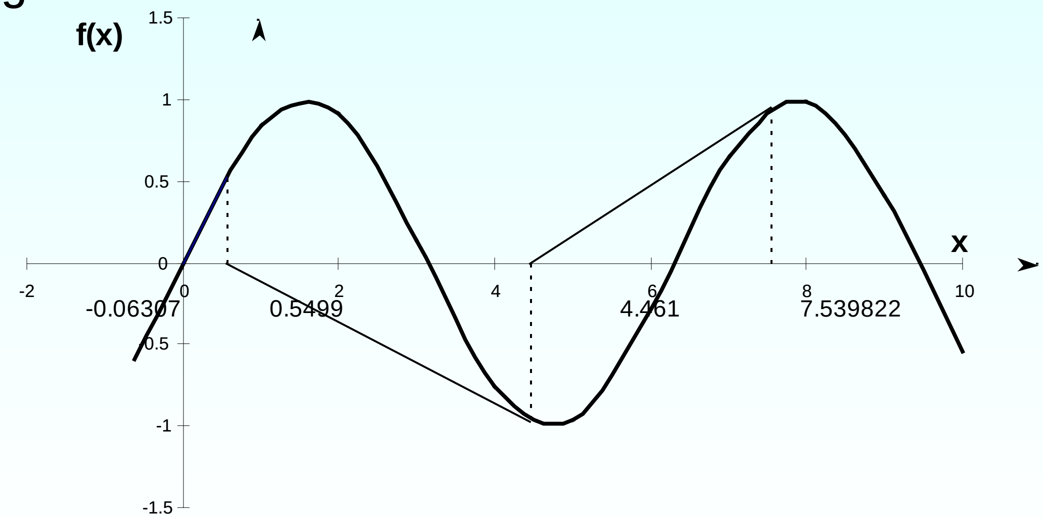

Figure 11 Root jumping from intended location of root for $f(x) = \sin x = 0$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/newton_raphson.html

THE END

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