Chapter 04.??

Effect of Significant Digits
on
Solution Of Simultaneous Linear Equations?

What is the effect of significant digits on solution of simultaneous linear equations?

Let us answer this question through an example where we solve a set of simultaneous linear equations by Naïve Gaussian Elimination method, assuming that we have a computer that uses 1) six significant digits with chopping and then do the same problem by using 2) five significant digits with chopping.

Example

Use Naïve Gauss Elimination to solve

$$10x_1 - 7x_2 = 7$$

$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

Use <u>six</u> significant digits with chopping in your calculations.

Solution

Working in the matrix form

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

<u>Forward Elimination of Unknowns</u>

Dividing Row 1 by 10 and multiplying by –3, that is, multiplying Row 1 by -0.3, and subtract it from Row 2 would eliminate a_{21},

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 6 \end{bmatrix}$$

Again dividing Row 1 by 10 and multiplying by 5, that is, multiplying Row 1 by 0.5, and subtract it from Row 3 would eliminate a_{31},

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

This is the end of the first step of forward elimination.

Now for the second step of forward elimination, we would use Row 2 as the pivot equation and eliminate Row 3 – Column 2. Dividing Row 2 by –0.001 and multiplying by 2.5, that is multiplying Row 2 by –2500, and subtracting from Row 3 gives

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15005 \end{bmatrix}$$

This is the end of the forward elimination steps.

<u>Back substitution</u>

We can now solve the above equations by back substitution. From the third equation,

$$15005x_3 = 15005$$

$$x_3 = \frac{15005}{15005}$$

$$= 1.$$

Substituting the value of x_3 in the second equation

$$-0.001x_2 + 6x_3 = 6.001$$

$$x_2 = \frac{6.001 - 6x_3}{-0.001}$$

$$= \frac{6.001 - 6(1)}{-0.001}$$

$$= \frac{6.001 - 6}{-0.001}$$

$$= \frac{0.001}{-0.001}$$

$$= -1$$

Substituting the value of x_3 and x_2 in the first equation,

$$10x_1 - 7x_2 + 0x_3 = 7$$

$$x_1 = \frac{7 + 7\,x_2 - 0x_3}{10}$$

$$= \frac{7 + 7(-1) - 0(1)}{10}$$

$$= 0$$

Hence the solution is

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Example

In the previous example we used Naïve Gauss Elimination to solve

$$10x_1 - 7x_2 = 7$$

$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

using six significant digits with chopping in your calculations. Repeat the problem, but now use five significant digits with chopping in your calculations.

Solution

Writing in the matrix form

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Forward Elimination of Unknowns

Dividing Row 1 by 10 and multiplying by –3, that is, multiplying Row 1 by -0.3, and subtract it from Row 2 would eliminate a_{21},

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 6 \end{bmatrix}$$

Again dividing Row 1 by 10 and multiplying by 5, that is, multiplying the Row 1 by 0.5, and subtract it from Row 3 would eliminate a_{31},

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

This is the end of the first step of forward elimination.

Now for the second step of forward elimination, we would use Row 2 as the pivoting equation and eliminate Row 3 – Column 2. Dividing Row 2 by –0.001 and multiplying by 2.5, that is, multiplying Row 2 by –2500, and subtract from Row 3 gives

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$

This is the end of the forward elimination steps.

<u>Back substitution</u>

We can now solve the above equations by back substitution. From the third equation,

$$15005x_3 = 15004$$

$$x_3 = \frac{15004}{15005}$$

$$= 0.99993$$

Substituting the value of x_3 in the second equation

$$-0.001x_2 + 6x_3 = 6.001$$

$$x_2 = \frac{6.001 - 6x_3}{-0.001}$$

$$= \frac{6.001 - 6(0.99993)}{-0.001}$$

$$= \frac{6.001 - 5.9995}{-0.001}$$

$$= \frac{0.0015}{-0.001}$$

$$= -1.5$$

Substituting the value of x_3 and x_2 in the first equation,

$$10x_1 - 7x_2 + 0x_3 = 7$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10}$$

$$= \frac{7 + 7(-1.5) - 0(1)}{10}$$

$$= \frac{7 - 10.5 - 0}{10}$$

$$= \frac{-3.5}{10}$$

$$= -0.3500$$

Hence the solution is

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -0.35 \\ -1.5 \\ 0.99993 \end{bmatrix}$$

Compare this with the exact solution of

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

This difference is a result of round off error caused by using only five significant digits in our calculations. The round off error is generally reduced by using Gaussian elimination with partial pivoting.

In the previous two examples, we used Naïve Gauss Elimination to solve

$$10x_1 - 7x_2 = 7$$

$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

using five and six significant digits with chopping in the calculations. Using <u>five</u> significant digits with chopping, the solution found was

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -0.35 \\ -1.5 \\ 0.99993 \end{bmatrix}$$

This is different from the exact solution

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Find the solution using Gaussian elimination with partial pivoting using five significant digits with chopping in your calculations.

Solution

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.901 \\ 6 \end{bmatrix}$$

Forward Elimination of Unknowns

Now for the first step of forward elimination, the absolute value of first column elements are

$$|10|, |-3|, |5|$$

or

10, 3, 5

So the largest absolute value is in the Row 1. So as per Gaussian Elimination with partial pivoting, the switch is between Row 1 and Row 1 to give

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Dividing Row 1 by 10 and multiplying by –3, that is, multiplying the Row 1 by -0.3, and subtract it from Row 2 would eliminate a_{21},

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 6 \end{bmatrix}$$

Again dividing Row 1 by 10 and multiplying by 5, that is, multiplying the Row 1 by 0.5, and subtract it from Row 3 would eliminate a_{31},

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

This is the end of the first step of forward elimination.

Now for the second step of forward elimination, the absolute value of the second column elements below the Row 2 is

$$|-0.001|, \ |2.5|$$

or

0.001, 2.5

So the largest absolute value is in Row 3. So the Row 2 is switched with the Row 3 to give

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$

Dividing row 2 by 2.5 and multiplying by –0.001, that is multiplying by 0.001/2.5=-0.0004, and then subtracting from Row 3 gives

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

<u>Back substitution</u>

$$6.002x_3 = 6.002$$

$$x_3 = \frac{6.002}{6.002}$$

$$=1$$

Substituting the value of x_3 in Row 2

$$2.5x_2 + 5x_3 = 2.5$$

$$x_2 = \frac{2.5 - 5x_2}{2.5}$$

$$= \frac{2.5 - 5(1)}{2.5}$$

$$= \frac{2.5 - 5}{2.5}$$

$$= \frac{-2.5}{2.5}$$

$$= -1$$

Substituting the value of x_3 and x_2 in Row 1

$$10x_1 - 7x_2 + 0x_2 = 7$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10}$$

$$= \frac{7 + 7(-1) - 0(1)}{10}$$

$$= \frac{7-7-0}{10}$$

$$= \frac{0}{10}$$

$$= 0$$

So the solution is

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

This, in fact, is the exact solution. By coincidence only, in this case, the round off error is fully removed.