LU Decomposition

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LU Decomposition

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LU Decomposition

LU Decomposition is another method to solve a set of simultaneous linear equations

Which is better, Gauss Elimination or LU Decomposition?

To answer this, a closer look at LU decomposition is needed.

LU Decomposition

Method

For most non-singular matrix [*A*] that one could conduct Naïve Gauss Elimination forward elimination steps, one can always write it as

$$[A] = [L][U]$$

where

$[L]$ = lower triangular matrix

$[U]$ = upper triangular matrix

How does LU Decomposition work?

If solving a set of linear equations	$[A][X] = [C]$
If $[A] = [L][U]$ then	$[L][U][X] = [C]$
Multiply by	$[L]^{-1}$
Which gives	$[L]^{-1}[L][U][X] = [L]^{-1}[C]$
Remember $[L]^{-1}[L] = [I]$ which leads to	$[I][U][X] = [L]^{-1}[C]$
Now, if $[I][U] = [U]$ then	$[U][X] = [L]^{-1}[C]$
Now, let	$[L]^{-1}[C]=[Z]$
Which ends with	$[L][Z] = [C] \quad (1)$
and	$[U][X] = [Z] \quad (2)$

LU Decomposition

How can this be used?

Given $[A][X] = [C]$

1. Decompose $[A]$ into $[L]$ and $[U]$
2. Solve $[L][Z] = [C]$ for $[Z]$
3. Solve $[U][X] = [Z]$ for $[X]$

When is LU Decomposition better than Gaussian Elimination?

To solve $[A][X] = [B]$

Table. Time taken by methods

Gaussian Elimination	LU Decomposition
$T\left(\frac{8n^3}{3}+12n^2+\frac{4n}{3}\right)$	$T\left(\frac{8n^3}{3}+12n^2+\frac{4n}{3}\right)$

where T = clock cycle time and n = size of the matrix

So both methods are equally efficient.

To find inverse of [A]

Time taken by Gaussian Elimination

$$= n\left(CT|_{FE} + CT|_{BS}\right)$$

$$= T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right)$$

Time taken by LU Decomposition

$$= CT|_{LU} + n \times CT|_{FS} + n \times CT|_{BS}$$

$$= T\left(\frac{32n^3}{3} + 12n^2 + \frac{20n}{3}\right)$$

Table 1 Comparing computational times of finding inverse of a matrix using LU decomposition and Gaussian elimination.

n	10	100	1000	10000
$CT\|_{\text{inverse GE}} / CT\|_{\text{inverse LU}}$	3.28	25.83	250.8	2501

Method: [A] Decompose to [L] and [U]

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

[*U*] is the same as the coefficient matrix at the end of the forward elimination step.

[*L*] is obtained using the *multipliers* that were used in the forward elimination process

Finding the [*U*] matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Step 1: $\frac{64}{25} = 2.56; \quad Row2 - Row1(2.56) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$

$$\frac{144}{25} = 5.76; \quad Row3 - Row1(5.76) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Finding the [U] Matrix

Matrix after Step 1: $\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$

Step 2: $\frac{-16.8}{-4.8} = 3.5; \quad Row3 - Row2(3.5) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$

$$[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Finding the [L] matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

Using the multipliers used during the Forward Elimination Procedure

From the first step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\ell_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$\ell_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

Finding the [L] Matrix

From the second step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} \qquad \ell_{32} = \frac{a_{32}}{a_{22}} = \frac{-16.8}{-4.8} = 3.5$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Does [L][U] = [A]?

$$[L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} = \quad ?$$

Example: Cylinder Stresses

To find the maximum stresses in a compound cylinder, the following four simultaneous linear equations need to solved.

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 4.2857\times10^{7} & -5.4619\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 0 \\ 0.007 \\ 0 \end{bmatrix}$$

Example: Cylinder Stresses

In the compound cylinder, the inner cylinder has an internal radius of *a = 5"*, and outer radius *c = 6.5"*, while the outer cylinder has an internal radius of *c = 6.5"* and outer radius, *b=8"*. Given E = 30×10^6 psi, ν = 0.3, and that the hoop stress in outer cylinder is given by

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[c_3(1+\nu)+c_4\left(\frac{1-\nu}{r^2}\right)\right]$$

find the stress on the inside radius of the outer cylinder.
Find the values of c_1, c_2, c_3 and c_4 using LU decomposition.

Example: Cylinder Stresses

Use Forward Elimination Procedure of Gauss Elimination to find [U]

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 4.2857\times10^{7} & -5.4619\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix}$$

Step 1

$$\frac{4.2857\times10^{7}}{4.2857\times10^{7}}=1; \quad Row2-Row1(1)=$$

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix}$$

Example: Cylinder Stresses

Step 1 cont.

$$\frac{-6.5}{4.2857\times10^{7}} = -1.5167\times10^{-7}; \quad Row3 - Row1\left(-1.5167\times10^{-7}\right) =$$

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ 0 & -0.29384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix}$$

Example: Cylinder Stresses

Step 1 cont.

$$\frac{0}{4.2857\times10^{7}}=0;\quad Row4-Row1(0)=$$

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ 0 & -0.29384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix}$$

This is the matrix after Step 1.

Example: Cylinder Stresses

Step 2

$$\frac{-0.29384}{3.7688\times 10^{5}} = -7.7966\times 10^{-7}; \quad Row3 - Row2\left(-7.7966\times 10^{-7}\right) =$$

$$\begin{bmatrix} 4.2857\times 10^{7} & -9.2307\times 10^{5} & 0 & 0 \\ 0 & 3.7688\times 10^{5} & -4.2857\times 10^{7} & 5.4619\times 10^{5} \\ 0 & 0 & -26.914 & 0.57968 \\ 0 & 0 & 4.2857\times 10^{7} & -3.6057\times 10^{5} \end{bmatrix}$$

Example: Cylinder Stresses

Step 2 cont.

$$\frac{0}{3.7688\times10^{5}}=0;\quad Row4-Row2(0)=$$

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ 0 & 0 & -26.914 & 0.57968 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix}$$

This is the matrix after Step 2.

Example: Cylinder Stresses

Step 3

$$\frac{4.2857\times10^{7}}{-26.914}=-1.5294\times10^{6};\quad Row4-Row3\left(-1.5923\times10^{6}\right)=$$

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ 0 & 0 & -26.914 & 0.57968 \\ 0 & 0 & 0 & 5.6250\times10^{5} \end{bmatrix}$$

This is the matrix after Step 3.

Example: Cylinder Stresses

$$[U] = \begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ 0 & 0 & -26.914 & 0.57968 \\ 0 & 0 & 0 & 5.6250\times10^{5} \end{bmatrix}$$

Example: Cylinder Stresses

Use the multipliers from Forward Elimination to find [L]

From the 1st step of forward elimination

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 4.2857\times10^{7} & -5.4619\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 \\ \ell_{31} & \ell_{32} & 1 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 \end{bmatrix}$$

$$\ell_{21} = \frac{a_{21}}{a_{11}} = \frac{4.2857\times10^{7}}{4.2857\times10^{7}} = 1$$

$$\ell_{31} = \frac{a_{31}}{a_{11}} = \frac{-6.5}{4.2857\times10^{7}} = -1.5167\times10^{-7}$$

$$\ell_{41} = \frac{a_{41}}{a_{11}} = \frac{0}{4.2857\times10^{7}} = 0$$

Example: Cylinder Stresses

From the 2nd step of forward elimination

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ 0 & -0.29384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 \\ \ell_{31} & \ell_{32} & 1 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 \end{bmatrix}$$

$$\ell_{32} = \frac{a_{32}}{a_{22}} \frac{-0.29384}{3.7688\times10^{5}} = -7.7966\times10^{-7}$$

$$\ell_{42} = \frac{a_{42}}{a_{22}} = \frac{0}{3.7688\times10^{5}} = 0$$

Example: Cylinder Stresses

From the 3rd step of forward elimination

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ 0 & 0 & -26.914 & 0.57968 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 \\ \ell_{31} & \ell_{32} & 1 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 \end{bmatrix}$$

$$\ell_{43} = \frac{a_{43}}{a_{33}} = \frac{4.2857\times10^{7}}{-26.914} = -1.5294\times10^{6}$$

Example: Cylinder Stresses

$$[L]=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1.5167\times10^{-7} & -7.7966\times10^{-7} & 1 & 0 \\ 0 & 0 & -1.5924\times10^{6} & 1 \end{bmatrix}$$

Example: Cylinder Stresses

Does [L][U] = [A]?

$$[L][U]=$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1.5167\times10^{-7} & -7.7966\times10^{-7} & 1 & 0 \\ 0 & 0 & -1.5924\times10^{6} & 1 \end{bmatrix}\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ 0 & 0 & -26.914 & 0.57968 \\ 0 & 0 & 0 & 5.6250\times10^{5} \end{bmatrix} = ?$$

Example: Cylinder Stresses

Set [L][Z] = [C]

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1.5167\times10^{-7} & -7.7966\times10^{-7} & 1 & 0 \\ 0 & 0 & -1.5924\times10^{6} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 0 \\ 0.007 \\ 0 \end{bmatrix}$$

Solve for [Z]

$$z_1 = -7.887\times10^{3}$$

$$z_1 + z_2 = 0$$

$$-1.51667\times10^{-7} z_1 + \left(-7.79662\times10^{-7}\right) z_2 + z_3 = 0.007$$

$$-1.5924\times10^{6} z_3 + z_4 = 0$$

Example: Cylinder Stresses

Solve for [Z]

$$z_1 = -7.887 \times 10^3$$

$$z_2 = -z_1$$
$$= -\left(-7.887 \times 10^3\right)$$
$$= 7.887 \times 10^3$$

$$z_3 = 0.007 - \left(-1.5167 \times 10^{-7}\right)z_1 - \left(-7.7966 \times 10^{-7}\right)z_2$$
$$= 0.007 - \left(-1.5167 \times 10^{-7}\right) \times \left(-7.887 \times 10^3\right) - \left(-7.7966 \times 10^{-7}\right) \times \left(7.887 \times 10^3\right)$$
$$= 1.1953 \times 10^{-2}$$

Example: Cylinder Stresses

Solving for [Z] cont.

$$z_4 = -\left(-1.5924\times10^{6}\right)z_3$$

$$= -\left(-1.5924\times10^{6}\right)\times\left(1.1953\times10^{-2}\right)$$

$$= 19034$$

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 7.887\times10^{3} \\ 1.1953\times10^{-2} \\ 19034 \end{bmatrix}$$

Example: Cylinder Stresses

Set [U][C] = [Z]

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ 0 & 0 & -26.914 & 0.57968 \\ 0 & 0 & 0 & 5.6250\times10^{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 7.887\times10^{3} \\ 1.1953\times10^{-2} \\ 19034 \end{bmatrix}$$

The four equations become:

$$4.2857\times10^{7}c_1 + \left(-9.2307\times10^{5}\right)c_2 + \left(0\right)c_3 + \left(0\right)c_4 = -7.887\times10^{3}$$

$$3.7668\times10^{5}c_2 + \left(-4.2857\times10^{7}\right)c_3 + 5.4619\times10^{5}c_4 = 7.887\times10^{3}$$

$$-26.914c_3 + 0.57968c_4 = 1.1953\times10^{-2}$$

$$5.6250\times10^{5}c_4 = 19034$$

Example: Cylinder Stresses

Solve for [C]

$$c_4 = \frac{19034}{5.6250 \times 10^5}$$

$$= 3.3837 \times 10^{-2}$$

$$c_3 = \frac{1.1953 \times 10^{-2} - 0.57968 c_4}{-26.914}$$

$$= \frac{1.1953 \times 10^{-2} - 0.57968 c_4}{-26.914}$$

$$= \frac{1.1953 \times 10^{-2} - 0.57968 \left(3.3837 \times 10^{-2}\right)}{-26.914}$$

$$= 2.8469 \times 10^{-4}$$

Example: Cylinder Stresses

Solve for [C] cont.

$$c_2 = \frac{7.887\times10^{3} - \left(-4.2857\times10^{7}\right)c_3 - 5.4619\times10^{5}c_4}{3.7688\times10^{5}}$$

$$= \frac{7.887\times10^{3} - \left(-4.2857\times10^{7}\right)\times\left(2.8469\times10^{-4}\right) - 5.4619\times10^{5}\times\left(3.3837\times10^{-2}\right)}{3.7688\times10^{5}}$$

$$= 4.2615\times10^{-3}$$

$$c_1 = \frac{-7.887\times10^{3} - \left(-9.2307\times10^{5}\right)c_2 - (0)c_3 \text{-} (0)c_4}{4.2857\times10^{7}}$$

$$= \frac{-7.887\times10^{3} - \left(-9.2307\times10^{5}\right)\times\left(4.2615\times10^{-3}\right)}{4.2857\times10^{7}}$$

$$= 9.2244\times10^{-5}$$

Example: Cylinder Stresses

Solution:

The solution vector is

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -9.2244\times10^{-5} \\ 4.2615\times10^{-3} \\ 2.8469\times10^{-4} \\ 3.3837\times10^{-2} \end{bmatrix}$$

The stress on the inside radius of the outer cylinder is then given by

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[c_3(1+\nu)+c_4\left(\frac{1-\nu}{r^2}\right)\right]$$

$$= \frac{30\times10^6}{1-0.3^2}\left[2.8469\times10^{-4}(1+0.3)+3.3837\times10^{-2}\left(\frac{1-0.3}{6.5^2}\right)\right]$$

$$= 30683 \text{ psi}$$

Finding the inverse of a square matrix

The inverse [B] of a square matrix [*A*] is defined as

$$[A][B] = [I] = [B][A]$$

Finding the inverse of a square matrix

How can LU Decomposition be used to find the inverse?

Assume the first column of [*B*] to be $[b_{11}\ \ b_{12}\ \ \ldots\ b_{n1}]^T$

Using this and the definition of matrix multiplication

First column of [*B*]

$$[A]\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Second column of [*B*]

$$[A]\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

The remaining columns in [*B*] can be found in the same manner

Example: Inverse of a Matrix

Find the inverse of a square matrix [*A*]

$$[A]=\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Using the decomposition procedure, the [*L*] and [*U*] matrices are found to be

$$[A]=[L][U]=\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Example: Inverse of a Matrix

Solving for the each column of [*B*] requires two steps

1) Solve [*L*] [*Z*] = [*C*] for [*Z*]

2) Solve [*U*] [*X*] = [*Z*] for [*X*]

Step 1: $$[L][Z]=[C]\rightarrow\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

This generates the equations:

$$z_1=1$$

$$2.56z_1+z_2=0$$

$$5.76z_1+3.5z_2+z_3=0$$

Example: Inverse of a Matrix

Solving for [*Z*]

$$z_1 = 1$$

$$\begin{aligned} z_2 &= 0 - 2.56 z_1 \\ &= 0 - 2.56(1) \\ &= -2.56 \end{aligned}$$

$$\begin{aligned} z_3 &= 0 - 5.76 z_1 - 3.5 z_2 \\ &= 0 - 5.76(1) - 3.5(-2.56) \\ &= 3.2 \end{aligned}$$

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

Example: Inverse of a Matrix

Solving [*U*][*X*] = [*Z*] for [*X*]

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$25b_{11} + 5b_{21} + b_{31} = 1$$

$$-4.8b_{21} - 1.56b_{31} = -2.56$$

$$0.7b_{31} = 3.2$$

Example: Inverse of a Matrix

Using Backward Substitution

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$b_{21} = \frac{-2.56 + 1.560 b_{31}}{-4.8}$$

$$= \frac{-2.56 + 1.560(4.571)}{-4.8} = -0.9524$$

$$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25}$$

$$= \frac{1 - 5(-0.9524) - 4.571}{25} = 0.04762$$

So the first column of the inverse of [*A*] is:

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Example: Inverse of a Matrix

Repeating for the second and third columns of the inverse

Second Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

Third Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Example: Inverse of a Matrix

The inverse of [*A*] is

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

To check your work do the following operation

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/lu_decomposition.html

THE END

http://numericalmethods.eng.usf.edu