Spline Interpolation Method

Civil Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

Spline Method of Interpolation

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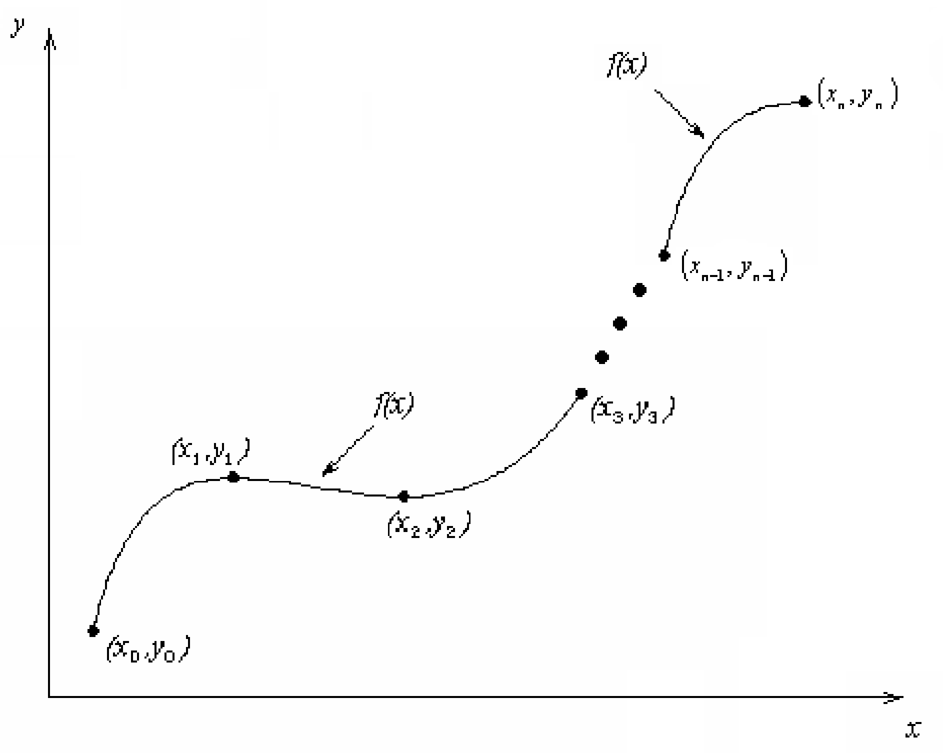
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Why Splines ?

$$f(x) = \frac{1}{1+25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

x	$y = \frac{1}{1+25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

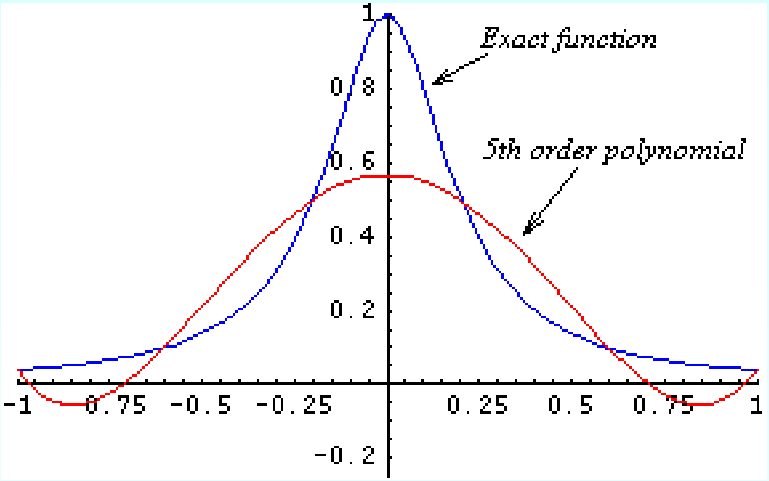


Figure : 5^{th} order polynomial vs. exact function

Why Splines ?

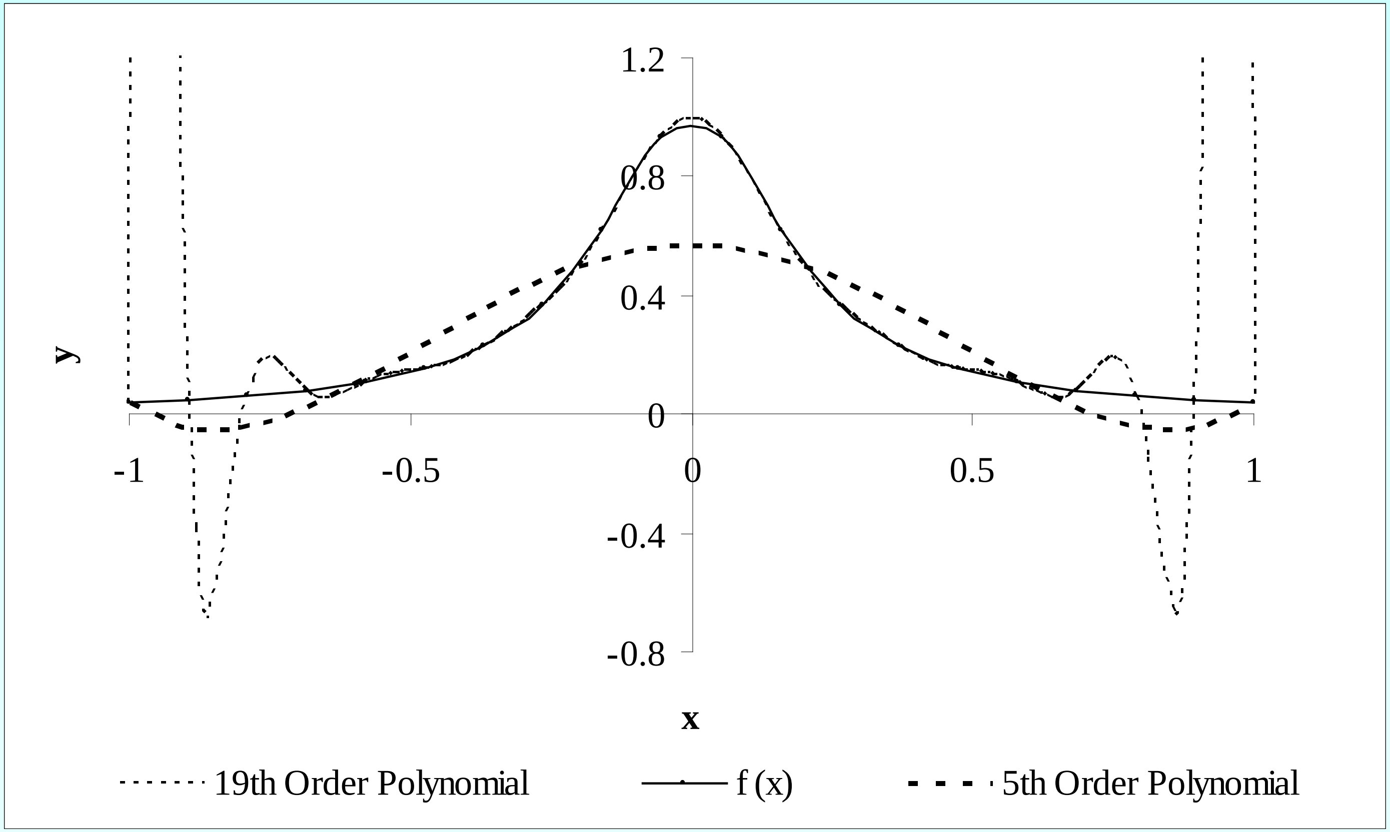
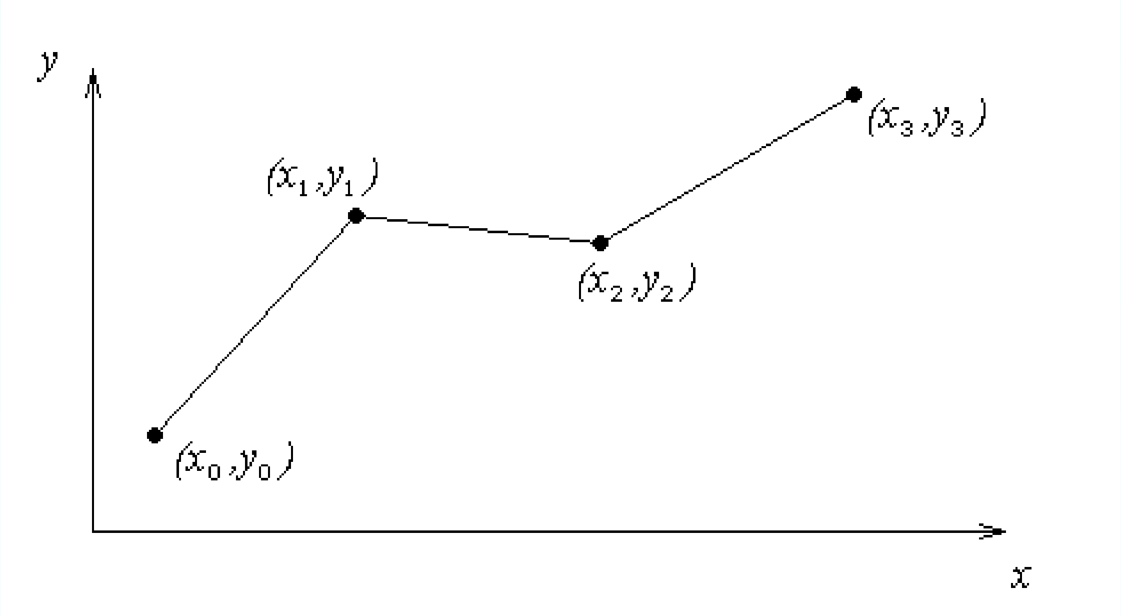


Figure : Higher order polynomial interpolation is a bad idea

Linear Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1})(x_n, y_n)$, fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $(y_i = f(x_i))$

Figure : Linear splines



Linear Interpolation (contd)

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2$$

$$.$$

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$$.$$

$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n$$

Note the terms of

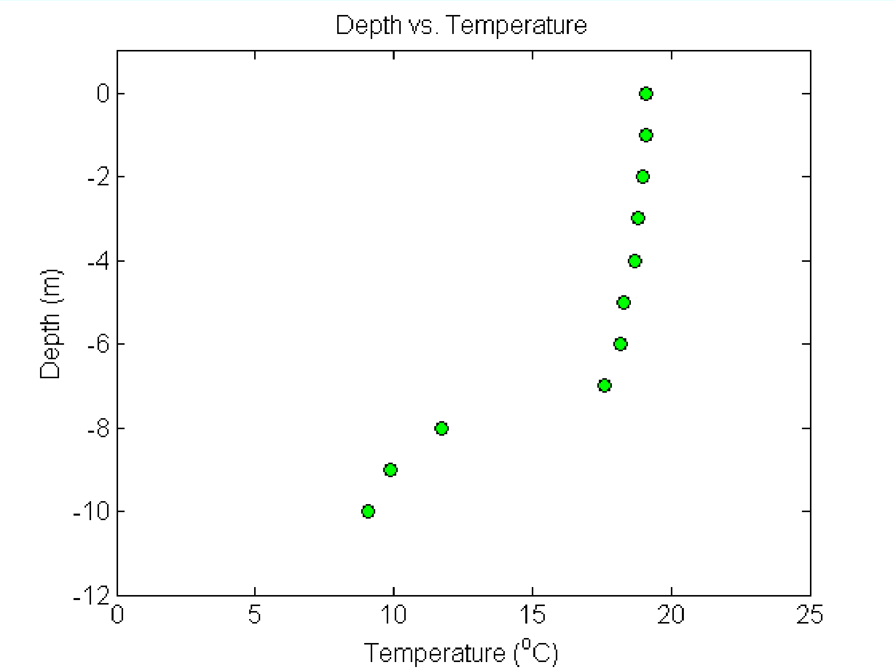
$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between x_{i-1} and x_i.

Example

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth plot for a lake. Determine the value of the temperature at $z = -7.5$ using Linear Spline Interpolation.

Temperature	Depth
T (^oC)	z (m)
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10



Temperature vs. depth of a lake

Linear Interpolation

$$z_0 = -8, \qquad T(z_0) = 11.7$$

$$z_1 = -7, \qquad T(z_1) = 17.6$$

$$T(z) = T(z_0) + \frac{T(z_1) - T(z_0)}{z_1 - z_0}(z - z_0)$$

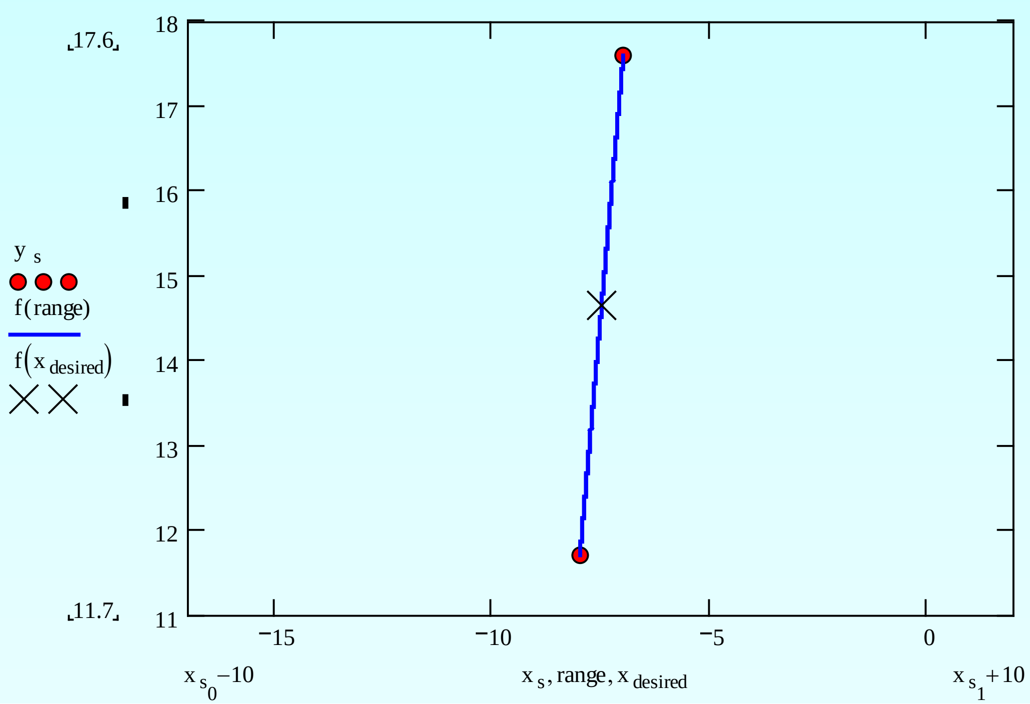
$$= 11.7 + \frac{17.6 - 11.7}{-7 + 8}(z + 8)$$

$$T(z) = 11.7 + 5.9(z + 8), \quad -8 \le z \le -7$$

At $z = -7.5$,

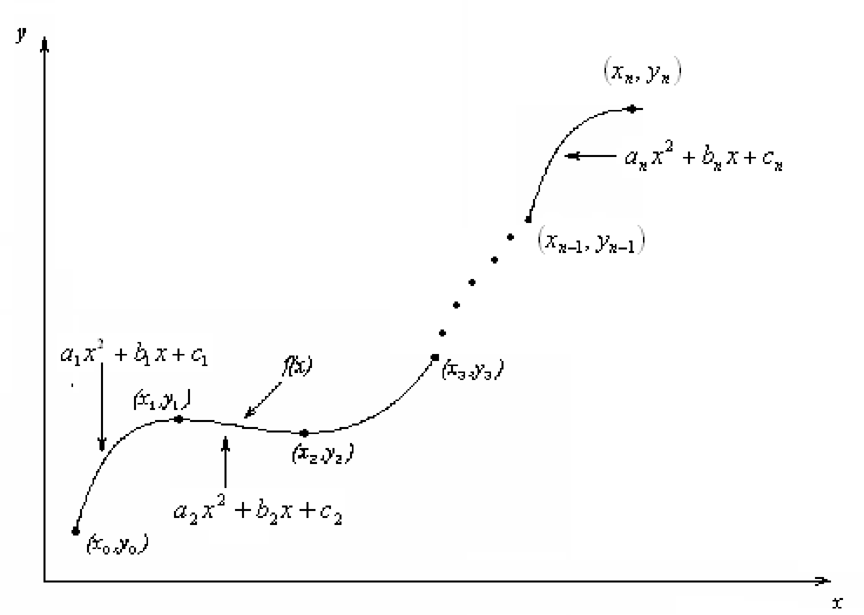
$$T(-7.5) = 11.7 + 5.9(-7.5 + 8)$$

$$= 14.65°C$$



Quadratic Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by



$$
\begin{aligned}
f(x) &= a_1 x^2 + b_1 x + c_1, && x_0 \le x \le x_1 \\
&= a_2 x^2 + b_2 x + c_2, && x_1 \le x \le x_2 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= a_n x^2 + b_n x + c_n, && x_{n-1} \le x \le x_n
\end{aligned}
$$

Find $a_i, b_i, c_i, i = 1, 2, \ldots,$ n

Quadratic Interpolation (contd)

Each quadratic spline goes through two consecutive data points

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$

$$a_1 x_1^2 + b_1 x_1 + c_1 = f(x_1)$$

.

.

$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

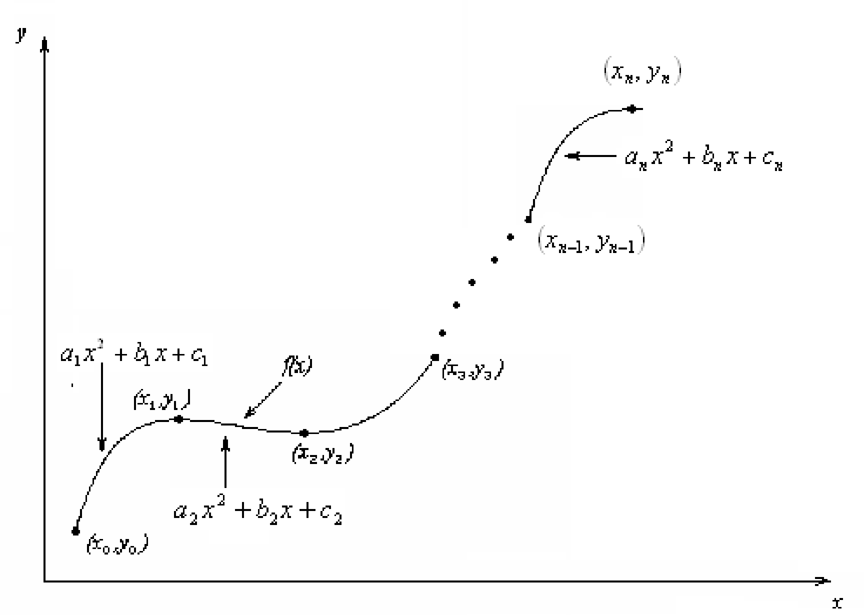
$$a_i x_i^2 + b_i x_i + c_i = f(x_i)$$

.

.

$$a_n x_{n-1}^2 + b_n x_{n-1} + c_n = f(x_{n-1})$$

$$a_n x_n^2 + b_n x_n + c_n = f(x_n)$$



This condition gives 2n equations

Quadratic Splines (contd)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1 \quad \text{is} \qquad 2a_1x + b_1$$

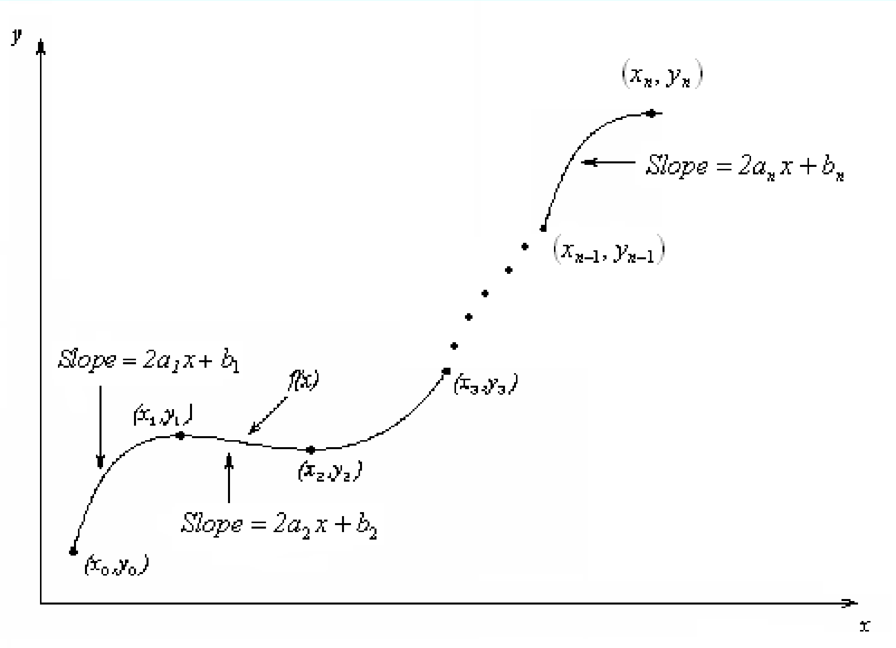
The derivative of the second spline

$$a_2x^2 + b_2x + c_2 \quad \text{is} \qquad 2a_2x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



Quadratic Splines (contd)

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

.

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$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

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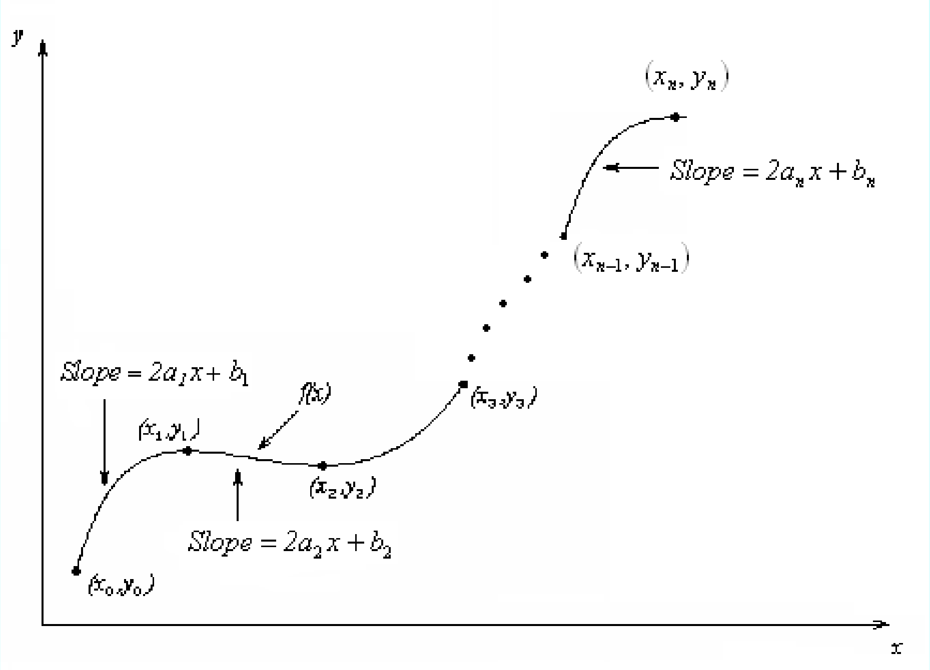
$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$



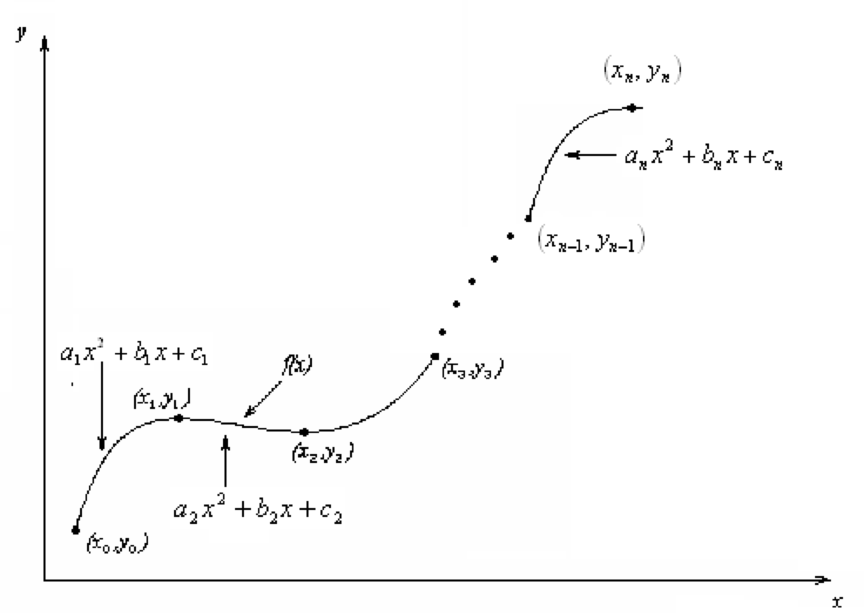
We have (n-1) such equations. The total number of equations is $(2n)+(n-1)=(3n-1)$.

We can assume that the first spline is linear, that is $a_1 = 0$

Quadratic Splines (contd)

This gives us ‘3n’ equations and ‘3n’ unknowns. Once we find the ‘3n’ constants, we can find the function at any value of ‘x’ using the splines,

$$
\begin{aligned}
f(x) &= a_1x^2 + b_1x + c_1, && x_0 \le x \le x_1 \\
&= a_2x^2 + b_2x + c_2, && x_1 \le x \le x_2 \\
&\;. \\
&\;. \\
&\;. \\
&= a_nx^2 + b_nx + c_n, && x_{n-1} \le x \le x_n
\end{aligned}
$$



Example

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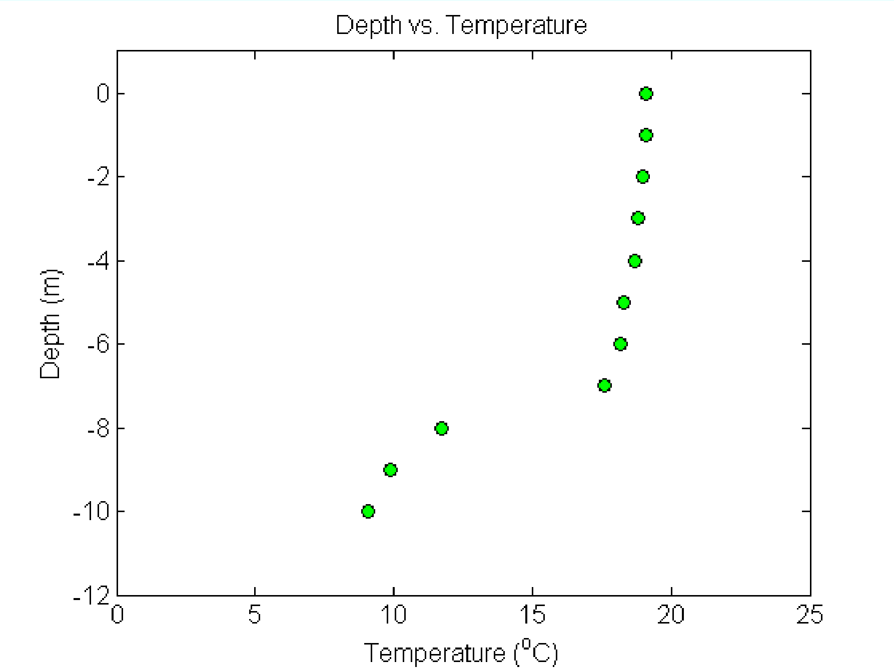
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10

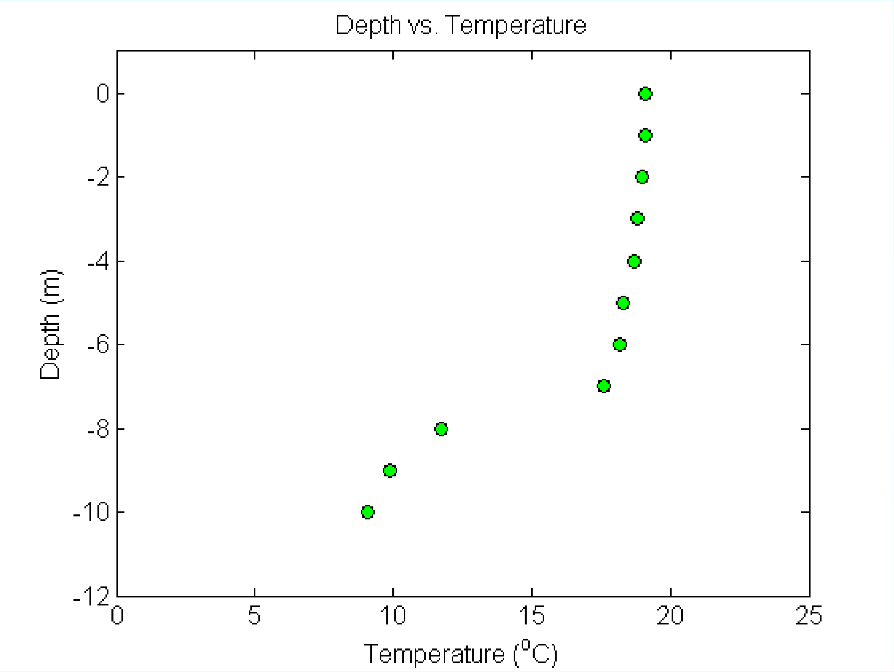


Temperature vs. depth of a lake

Solution

Since there are eleven data points,
ten quadratic splines pass through them.

$$
\begin{aligned}
T(z) &= a_1 z^2 + b_1 z + c_1, & -10 \le z \le -9 \\
&= a_2 z^2 + b_2 z + c_2, & -9 \le z \le -8 \\
&= a_3 z^2 + b_3 z + c_3, & -8 \le z \le -7 \\
&= a_4 z^2 + b_4 z + c_4, & -7 \le z \le -6 \\
&= a_5 z^2 + b_5 z + c_5, & -6 \le z \le -5 \\
&= a_6 z^2 + b_6 z + c_6, & -5 \le z \le -4 \\
&= a_7 z^2 + b_7 z + c_7, & -4 \le z \le -3 \\
&= a_8 z^2 + b_8 z + c_8, & -3 \le z \le -2 \\
&= a_9 z^2 + b_9 z + c_9, & -2 \le z \le -1 \\
&= a_{10} z^2 + b_{10} z + c_{10}, & -1 \le z \le 0
\end{aligned}
$$



Solution (contd)

Setting up the equations

Each quadratic spline passes through two consecutive data points giving

$a_1 z^2 + b_1 z + c_1$ passes through z = −10 and z = −9,

$$a_1(-10)^2 + b_1(-10) + c_1 = 9.1 \qquad (1)$$

$$a_1(-9)^2 + b_1(-9) + c_1 = 9.9 \qquad (2)$$

Similarly,

$$a_2(-9)^2 + b_2(-9) + c_2 = 9.9 \qquad (3)$$

$$a_2(-8)^2 + b_2(-8) + c_2 = 11.7 \qquad (4)$$

$$a_3(-8)^2 + b_3(-8) + c_3 = 11.7 \qquad (5)$$

$$a_3(-7)^2 + b_3(-7) + c_3 = 17.6 \qquad (6)$$

$$a_4(-7)^2 + b_4(-7) + c_4 = 17.6 \qquad (7)$$

$$a_4(-6)^2 + b_4(-6) + c_4 = 18.2 \qquad (8)$$

$$a_5(-6)^2 + b_5(-6) + c_5 = 18.2 \qquad (9)$$

$$a_5(-5)^2 + b_5(-5) + c_5 = 18.3 \qquad (10)$$

Solution (contd)

$$a_6(-5)^2 + b_6(-5) + c_6 = 18.3 \qquad (11)$$

$$a_6(-4)^2 + b_6(-4) + c_6 = 18.7 \qquad (12)$$

$$a_7(-4)^2 + b_7(-4) + c_7 = 18.7 \qquad (13)$$

$$a_7(-3)^2 + b_7(-3) + c_7 = 18.8 \qquad (14)$$

$$a_8(-3)^2 + b_8(-3) + c_8 = 18.8 \qquad (15)$$

$$a_8(-2)^2 + b_8(-2) + c_8 = 19 \qquad (16)$$

$$a_9(-2)^2 + b_9(-2) + c_9 = 19 \qquad (17)$$

$$a_9(-1)^2 + b_9(-1) + c_9 = 19.1 \qquad (18)$$

$$a_{10}(-1)^2 + b_{10}(-1) + c_{10} = 19.1 \qquad (19)$$

$$a_{10}(0)^2 + b_{10}(0) + c_{10} = 19.1 \qquad (20)$$

Solution (contd)

Quadratic splines have continuous derivatives at the interior data points

At $z=-9$ $\quad 2a_1(-9)+b_1-2a_2(-9)-b_2=0 \quad (21)$

At $z=-8$ $\quad 2a_2(-8)+b_2-2a_3(-8)-b_3=0 \quad (22)$

At $z=-7$ $\quad 2a_3(-7)+b_3-2a_4(-7)-b_4=0 \quad (23)$

At $z=-6$ $\quad 2a_4(-6)+b_4-2a_5(-6)-b_5=0 \quad (24)$

At $z=-5$ $\quad 2a_5(-5)+b_5-2a_6(-5)-b_6=0 \quad (25)$

At $z=-4$ $\quad 2a_6(-4)+b_6-2a_7(-4)-b_7=0 \quad (26)$

At $z=-3$ $\quad 2a_7(-3)+b_7-2a_8(-3)-b_8=0 \quad (27)$

At $z=-2$ $\quad 2a_8(-2)+b_8-2a_9(-2)-b_9=0 \quad (28)$

At $z=-1$ $\quad 2a_9(-1)+b_9-2a_{10}(-1)-b_{10}=0 \quad (29)$

Assuming the first spline $a_1z^2+b_1z+c_1$ is linear,

$$a_1=0 \qquad (30)$$

Solution (contd)

$$
\begin{bmatrix}
100 & -10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
81 & -9 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 81 & -9 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 64 & -8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 64 & -8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 49 & -7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 49 & -7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 36 & -6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 36 & -6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 & -5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-18 & 1 & 0 & 18 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -16 & 1 & 0 & 16 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -14 & 1 & 0 & 14 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 1 & 0 & 12 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 1 & 0 & 10 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 1 & 0 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 1 & 0 & 6 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 1 & 0 & 4 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 2 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \\ a_6 \\ b_6 \\ c_6 \\ a_7 \\ b_7 \\ c_7 \\ a_8 \\ b_8 \\ c_8 \\ a_9 \\ b_9 \\ c_9 \\ a_{10} \\ b_{10} \\ c_{10}
\end{bmatrix}
=
\begin{bmatrix}
9.1 \\ 9.9 \\ 9.9 \\ 11.7 \\ 11.7 \\ 17.6 \\ 17.6 \\ 18.2 \\ 18.2 \\ 18.3 \\ 18.3 \\ 18.7 \\ 18.7 \\ 18.8 \\ 18.8 \\ 19 \\ 19 \\ 19.1 \\ 19.1 \\ 19.1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

Solution (contd)

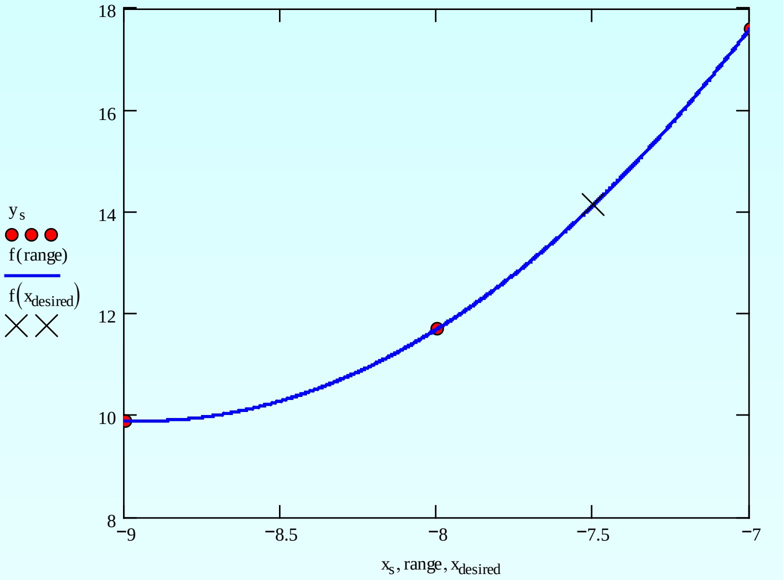
Solving the above 30 equations gives the 30 unknowns as

i	a_i	b_i	c_i
1	0	0.8	17.1
2	1	18.8	98.1
3	3.1	52.4	232.5
4	−8.4	−108.6	−331
5	7.9	87	255.8
6	−7.6	−68	−131.7
7	7.3	51.2	106.7
8	−7.2	−35.8	−23.8
9	7.1	21.4	33.4
10	−7.2	−7.2	19.1

Solution (contd)

Therefore, the splines are given by

$$
\begin{aligned}
T(z) &= 0.8z + 17.1, & -10 \le z \le -9 \\
&= z^2 + 18.8z + 98.1, & -9 \le z \le -8 \\
&= 3.1z^2 + 52.4z + 232.5, & -8 \le z \le -7 \\
&= -8.4z^2 - 108.6z - 331, & -7 \le z \le -6 \\
&= 7.9z^2 + 87z + 255.8, & -6 \le z \le -5 \\
&= -7.6z^2 - 68z + 131.7, & -5 \le z \le -4 \\
&= 7.3z^2 + 51.2z + 106.7, & -4 \le z \le -3 \\
&= -7.2z^2 - 35.8z - 23.8, & -3 \le z \le -2 \\
&= 7.1z^2 + 21.4z + 33.4, & -2 \le z \le -1 \\
&= -7.2z^2 - 7.2z + 19.1, & -1 \le z \le 0
\end{aligned}
$$



Solution (contd)

At z = −7.5

$$T(-7.5) = 3.1(-7.5)^2 + 52.4(-7.5) + 232.5$$

$$= 13.875°C$$

The absolute relative approximate error $\left|\in_a\right|$ obtained

between the results from the linear and quadratic splines is

$$\left|\in_a\right| = \left|\frac{13.875 - 14.65}{13.875}\right| \times 100$$

$$= 5.5856\%$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/spline_method.html

THE END

http://numericalmethods.eng.usf.edu