Runge 2^{nd} Order Method

Civil Engineering Majors

Authors: Autar Kaw, Charlie Barker

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Transforming Numerical Methods Education for STEM Undergraduates

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Runge-Kutta 2nd Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 2nd order method is given by

$$y_{i+1} = y_i + \left(a_1 k_1 + a_2 k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + p_1 h, y_i + q_{11} k_1 h\right)$$

Heun's Method

<u>Heun's method</u>

Here a_2=1/2 is chosen

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

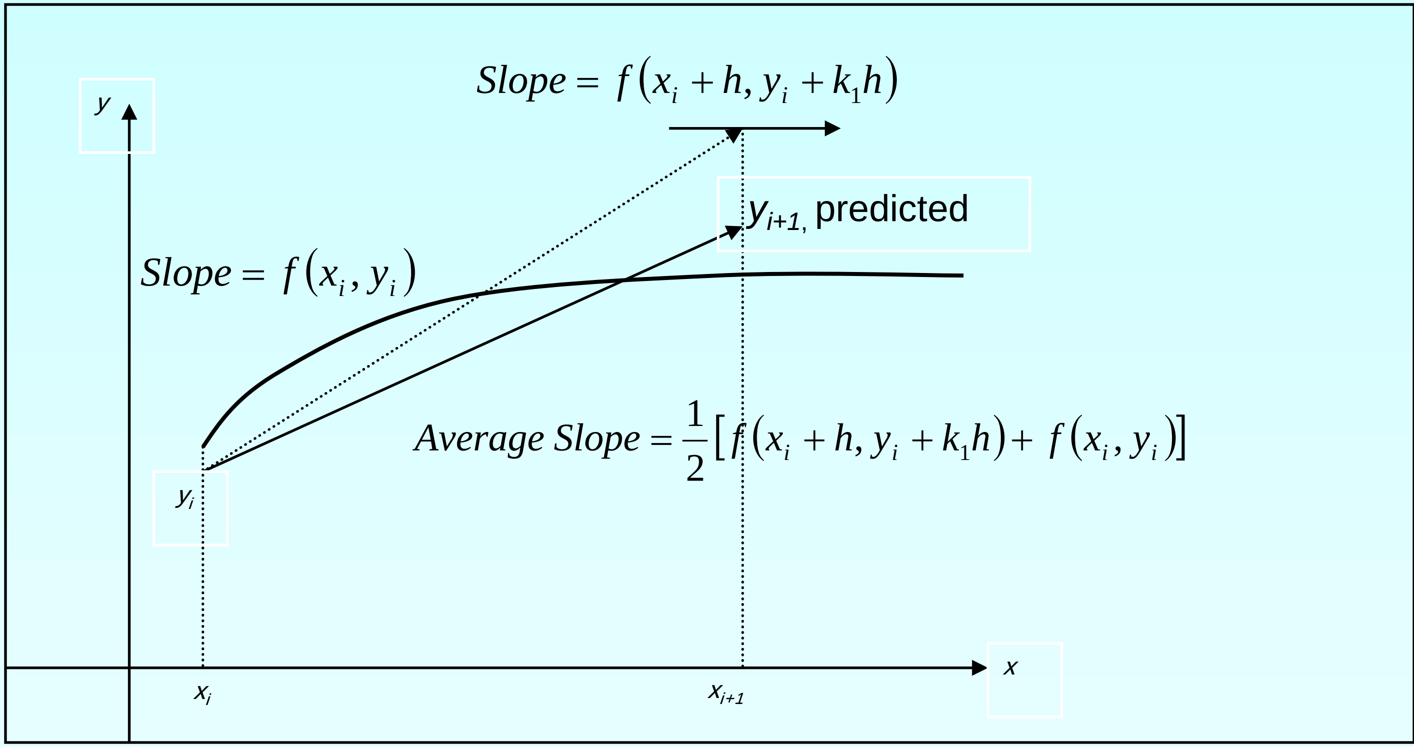

Figure 1 Runge-Kutta 2nd order method (Heun's method)

Midpoint Method

Here $a_2 = 1$ **is chosen, giving**

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

Ralston's Method

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A polluted lake with an initial concentration of a bacteria is 10^7 parts/m^3, while the acceptable level is only 5×10^6 parts/m^3. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration *C* of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt}+0.06C=0, C(0)=10^7$$

Find the concentration of the pollutant after 7 weeks. Take a step size of 3.5 weeks.

$$\frac{dC}{dt}=-0.06C$$

$$f\left(t,C\right)=-0.06C$$

$$C_{i+1}=C_i+\left(\frac{1}{2}k_1+\frac{1}{2}k_2\right)h$$

Solution

Step 1: $i = 0,\ t_0 = 0,\ C_0 = 10^7$

$$k_1 = f\left(t_0, C_0\right) = f\left(0, 10^7\right) = -0.06\left(10^7\right) = -600000$$

$$k_2 = f\left(t_0 + h,\ C_0 + k_1 h\right) = f\left(0 + 3.5,\ 10^7 + \left(-600000\right)3.5\right)$$
$$= f\left(3.5,\ 7.9\times10^6\right) = -0.06\left(7.9\times10^6\right) = -474000$$

$$C_1 = C_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$
$$= 10^7 + \left(\frac{1}{2}\left(-600000\right) + \frac{1}{2}\left(-474000\right)\right)3.5$$
$$= 10^7 + \left(-537000\right)3.5$$
$$= 8.1205\times10^6 \ \text{parts/m}^3$$

C_1 is the approximate concentration of bacteria at

$$t = t_1 = t_0 + h = 0 + 3.5 = 3.5\,\text{weeks}$$

$$C(3.5) \approx C_1 = 8.1205\times10^6 \ \text{parts/m}^3$$

Solution Cont

Step 2: $i=1,\ t_1 = t_0 + h = 0 + 3.5 = 3.5,\quad C_1 = 8.1205\times10^6 \text{ parts/m}^3$

$$k_1 = f(t_1, C_1) = f(3.5,\ 8.1205\times10^6) = -0.06(8.1205\times10^6) = -487230$$

$$k_2 = f(t_1 + h, C_1 + k_1 h) = f(3.5+3.5,\ 8.1205\times10^6 + (-487230)3.5)$$
$$= f(7,\ 6415200) = -0.06(6415200) = -384910$$

$$C_2 = C_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$
$$= 8.1205\times10^6 + \left(\frac{1}{2}(-487230) + \frac{1}{2}(-384910)\right)3.5$$
$$= 8.1205\times10^6 + (-436070)3.5$$
$$= 6.5943\times10^6 \text{ parts/m}^3$$

C_2 is the approximate concentration of bacteria at

$$t = t_2 = t_1 + h = 3.5 + 3.5 = 7 \text{ weeks}$$

$$C(7) \approx C_2 = 6.5943\times10^6 \text{ parts/m}^3$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$C(t) = 1\times 10^{7} e^{\left(\frac{-3t}{50}\right)}$$

The solution to this nonlinear equation at t=7 weeks is

$$C(7) = 6.5705\times 10^{6} \text{ parts/m}^{3}$$

Comparison with exact results

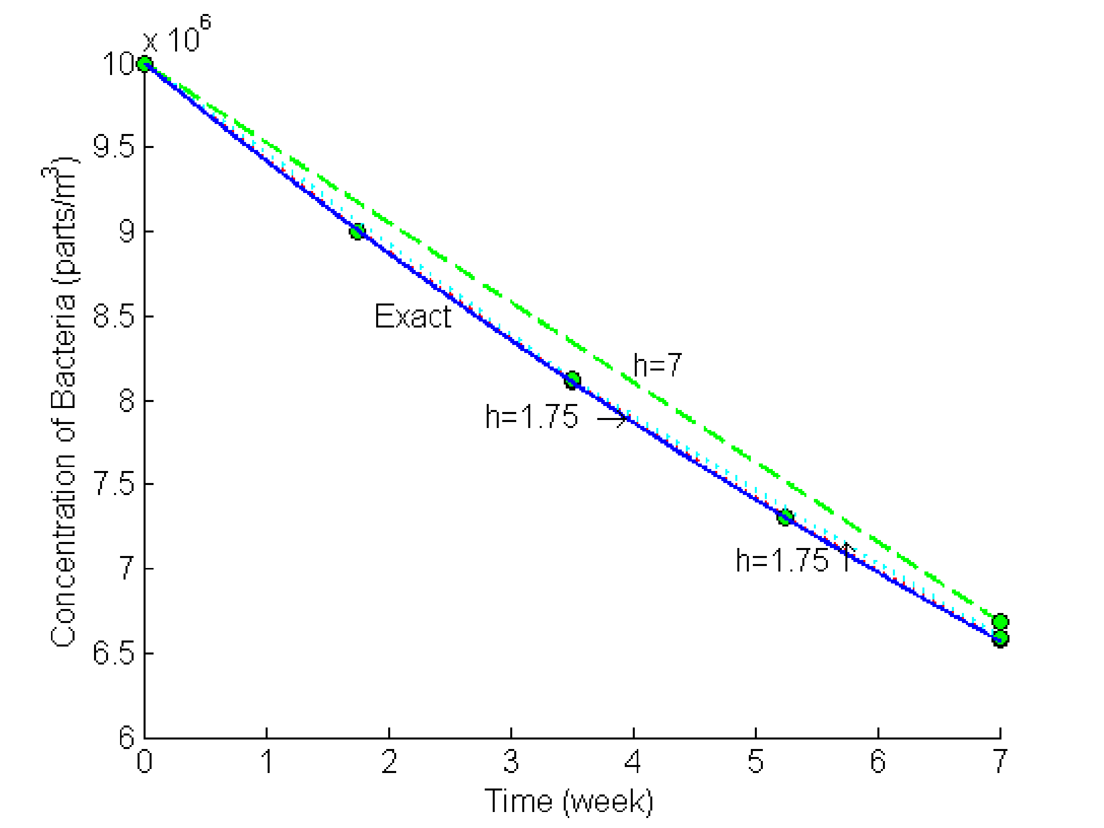

Figure 2. Heun's method results for different step sizes

Effect of step size

Table 1. Effect of step size for Heun's method

Step size, h	$C(7)$	E_t	$\lvert \in_t \rvert \%$
7	$6.6520 \quad 10^6$	-111530	1.6975
3.5	$6.5943 \quad 10^6$	-23784	0.36198
1.75	$6.5760 \quad 10^6$	-5489.1	0.083542
0.875	$6.5718 \quad 10^6$	-1318.8	0.020071
0.4375	$6.5708 \quad 10^6$	-323.24	0.0049195

$$C(7) = 6.5705 \times 10^6$$ (exact)

Effects of step size on Heun's Method

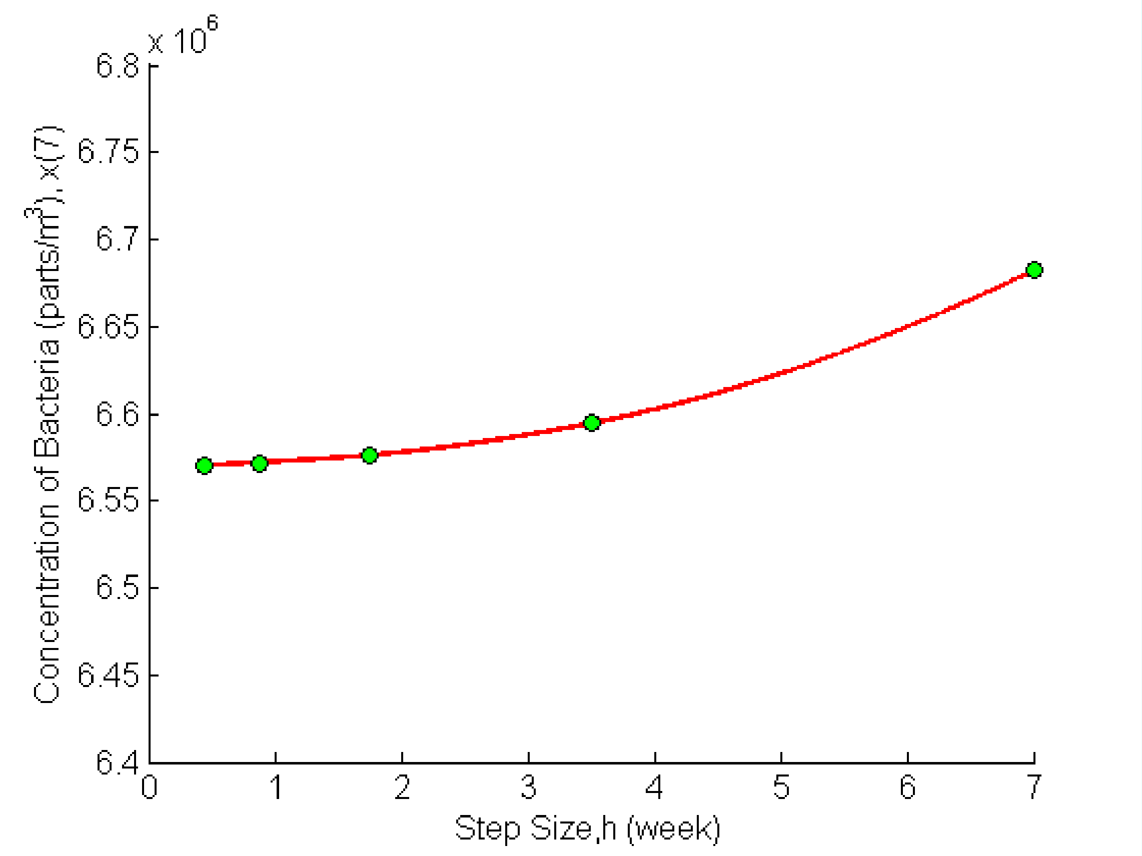

Figure 3. Effect of step size in Heun's method

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$C(7)$			
	Euler	Heun	Midpoint	Ralston
7	5.8000 10^6	6.6820 10^6	6.6820 10^6	6.6820 10^6
3.5	6.2410 10^6	6.5943 10^6	6.5943 10^6	6.5943 10^6
1.75	6.4160 10^6	6.5760 10^6	6.5760 10^6	6.5760 10^6
0.875	6.4960 10^6	6.5718 10^6	6.7518 10^6	6.5718 10^6
0.4375	6.5340 10^6	6.5708 10^6	6.5708 10^6	6.5708 10^6

$$C(7) = 6.5705 \times 10^6$$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\lvert \in_t \rvert \%$			
	Euler	Heun	Midpoint	Ralston
7	11.726	1.6975	1.6975	1.6975
3.5	5.0144	0.36198	0.36198	0.36198
1.75	2.3447	0.083542	0.083542	0.083542
0.875	1.1362	0.020071	0.020071	0.020071
0.4375	0.55952	0.0049195	0.0049195	0.0049195

$C(7) = 6.5705 \times 10^6$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

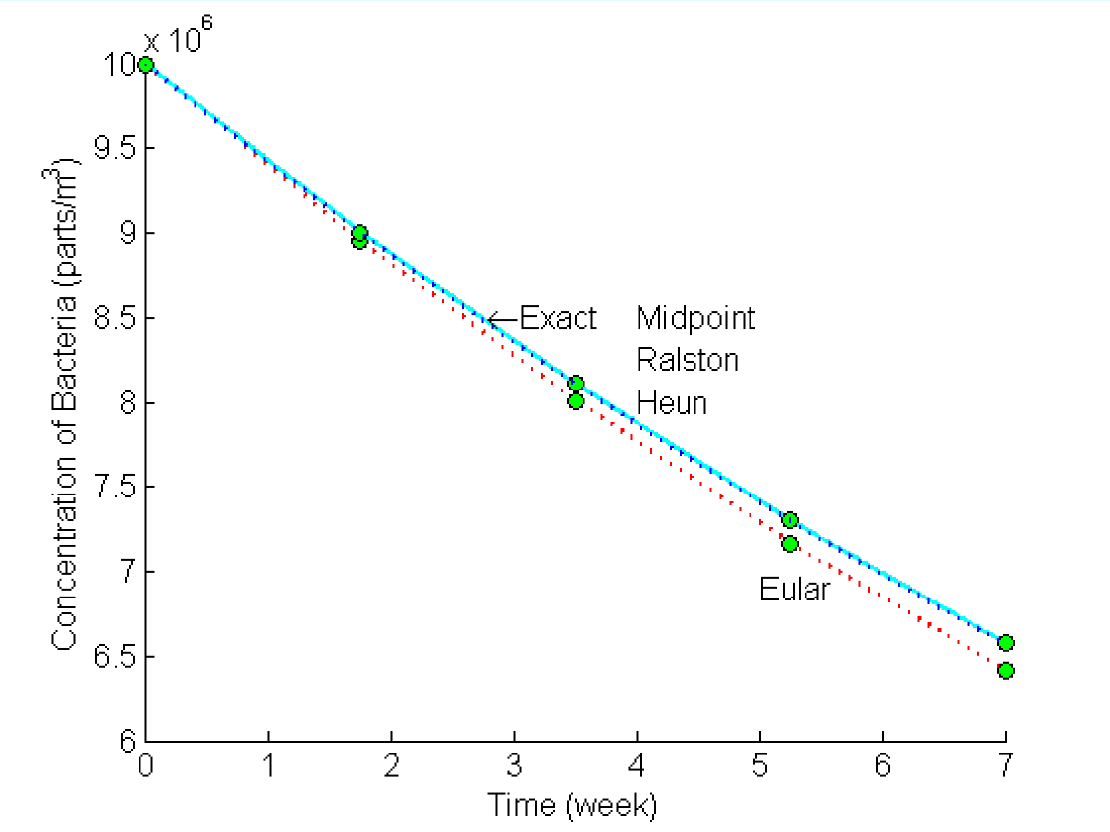

Figure 4. Comparison of Euler and Runge Kutta 2nd order methods with exact results.

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_2nd_method.html

THE END

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