Chapter 08.04
Runge-Kutta 4th Order Method for Ordinary Differential Equations

After reading this chapter, you should be able to

1. *develop Runge-Kutta 4^{th} order method for solving ordinary differential equations,*
2. *find the effect size of step size has on the solution,*
3. *know the formulas for other versions of the Runge-Kutta 4^{th} order method*

What is the Runge-Kutta 4th order method?

Runge-Kutta 4^{th} order method is a numerical technique used to solve ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

So only first order ordinary differential equations can be solved by using the Runge-Kutta 4^{th} order method. In other sections, we have discussed how Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \ y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx}+2y=1.3e^{-x},\ y(0)=5$$
$$\frac{dy}{dx}=1.3e^{-x}-2y,\ y(0)=5$$

In this case

$$f(x,y)=1.3e^{-x}-2y$$

Example 2

Rewrite

$$e^{y}\frac{dy}{dx}+x^{2}y^{2}=2\sin(3x),\ \ y(0)=5$$

in

$$\frac{dy}{dx}=f(x,y),\ \ y(0)=y_0 \text{ form.}$$

Solution

$$e^{y}\frac{dy}{dx}+x^{2}y^{2}=2\sin(3x),\ \ y(0)=5$$
$$\frac{dy}{dx}=\frac{2\sin(3x)-x^{2}y^{2}}{e^{y}},\ \ y(0)=5$$

In this case

$$f(x,y)=\frac{2\sin(3x)-x^{2}y^{2}}{e^{y}}$$

The Runge-Kutta 4th order method is based on the following

$$y_{i+1}=y_i+(a_1k_1+a_2k_2+a_3k_3+a_4k_4)h \tag{1}$$

where knowing the value of $y=y_i$ at x_i, we can find the value of $y=y_{i+1}$ at x_{i+1}, and

$$h=x_{i+1}-x_i$$

Equation (1) is equated to the first five terms of Taylor series

$$y_{i+1}=y_i+\frac{dy}{dx}\Big|_{x_i,y_i}(x_{i+1}-x_i)+\frac{1}{2!}\frac{d^2y}{dx^2}\Big|_{x_i,y_i}(x_{i+1}-x_i)^2+\frac{1}{3!}\frac{d^3y}{dx^3}\Big|_{x_i,y_i}(x_{i+1}-x_i)^3 +\frac{1}{4!}\frac{d^4y}{dx^4}\Big|_{x_i,y_i}(x_{i+1}-x_i)^4 \tag{2}$$

Knowing that $\frac{dy}{dx}=f(x,y)$ and $x_{i+1}-x_i=h$

$$y_{i+1}=y_i+f(x_i,y_i)h+\frac{1}{2!}f'(x_i,y_i)h^2+\frac{1}{3!}f''(x_i,y_i)h^3+\frac{1}{4!}f'''(x_i,y_i)h^4 \tag{3}$$

Based on equating Equation (2) and Equation (3), one of the popular solutions used is

$$y_{i+1}=y_i+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h \tag{4}$$

$$k_1 = f(x_i, y_i) \quad (5a)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) \quad (5b)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right) \quad (5c)$$

$$k_4 = f(x_i + h, y_i + k_3 h) \quad (5d)$$

Example 3

A polluted lake has an initial concentration of a bacteria of 10^7 parts/m^3, while the acceptable level is only 5×10^6 parts/m^3. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration C of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt} + 0.06C = 0, C(0) = 10^7$$

Using the Runge-Kutta 4th order method, find the concentration of the pollutant after 7 weeks. Take a step size of 3.5 weeks.

Solution

$$\frac{dC}{dt} = -0.06C$$

$$f(t, C) = -0.06C$$

$$C_{i+1} = C_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

For $i = 0$, $t_0 = 0$, $C_0 = 10^7$

$$\begin{aligned}
k_1 &= f(t_0, C_0) \\
&= f(0, 10^7) \\
&= -0.06(10^7) \\
&= -600000
\end{aligned}$$

$$\begin{aligned}
k_2 &= f\left(t_0 + \frac{1}{2}h, C_0 + \frac{1}{2}k_1 h\right) \\
&= f\left(0 + \frac{1}{2}\times 3.5, 10^7 + \frac{1}{2}(-600000)3.5\right) \\
&= f(1.75, 8950000) \\
&= -0.06(8950000) \\
&= -537000
\end{aligned}$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, C_0 + \frac{1}{2}k_2 h\right)$$

$$= f\left(0+\frac{1}{2}3.5, 10^7+\frac{1}{2}(-537000)3.5\right)$$
$$= f(1.75, 9060300)$$
$$= -0.06(9060300)$$
$$= -543620$$
$$k_4 = f(t_0+h, C_0+k_3h)$$
$$= f(0+3.5, 10^7+(-543620)3.5)$$
$$= f(3.5, 8097300)$$
$$= -0.06(8097300)$$
$$= -485840$$
$$C_1 = C_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$$
$$= 10^7+\frac{1}{6}(-600000+2(-537000)+2(-543620)+(-485840))3.5$$
$$= 10^7+\frac{1}{6}(-3247100)3.5$$
$$= 8.1059\times 10^6 \text{ parts/m}^3$$

C_1 is the approximate concentration of bacteria at

$$t = t_1 = t_0+h = 0+3.5 = 3.5 \text{ parts/m}^3$$
$$C(3.5) \approx C_1 = 8.1059\times 10^6 \text{ parts/m}^3$$

For $i = 1, t_1 = 3.5, C_1 = 8.1059\times 10^6$

$$k_1 = f(t_1, C_1)$$
$$= f(3.5, 8.1059\times 10^6)$$
$$= -0.06(8.1059\times 10^6)$$
$$= -486350$$
$$k_2 = f\left(t_1+\frac{1}{2}h, C_1+\frac{1}{2}k_1h\right)$$
$$= f\left(3.5+\frac{1}{2}\times 3.5, 8105900+\frac{1}{2}(-486350)3.5\right)$$
$$= f(5.25, 7254800)$$
$$= -0.06(7254800)$$
$$= -435290$$
$$k_3 = f\left(t_1+\frac{1}{2}h, C_1+\frac{1}{2}k_2h\right)$$
$$= f\left(3.5+\frac{1}{2}\times 3.5, 8105900+\frac{1}{2}(-435290)3.5\right)$$
$$= f(5.25, 7344100)$$
$$= -0.06(7344100)$$
$$= -440648$$

$$
\begin{aligned}
k_4 &= f(t_1 + h, C_1 + k_3 h) \\
&= f(3.5 + 3.5, 8105900 + (-440648)3.5) \\
&= f(7, 6563600) \\
&= -0.06(6563600) \\
&= -393820
\end{aligned}
$$

$$
\begin{aligned}
C_2 &= C_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\
&= 8105900 + \frac{1}{6}(-486350 + 2\times(-435290) + 2\times(-440648) + (-393820))\times 3.5 \\
&= 8105900 + \frac{1}{6}(-2632000)\times 3.5 \\
&= 6.5705\times 10^6 \text{ parts/m}^3
\end{aligned}
$$

C_2 is the approximate concentration of bacteria at

$$t_2 = t_1 + h = 3.5 + 3.5 = 7 \text{ weeks}$$

$$C(7) \approx C_2 = 6.5705\times 10^6 \text{ parts/m}^3$$

The exact solution of the ordinary differential equation is given by

$$C(t) = 1\times 10^7 e^{\left(\frac{-3t}{50}\right)}$$

The solution to this nonlinear equation at $t = 7$ weeks is

$$C(7) = 6.5705\times 10^6 \text{ parts/m}^3$$

Figure 1 compares the exact solution with the numerical solution using the Runge-Kutta 4th order method using different step sizes.

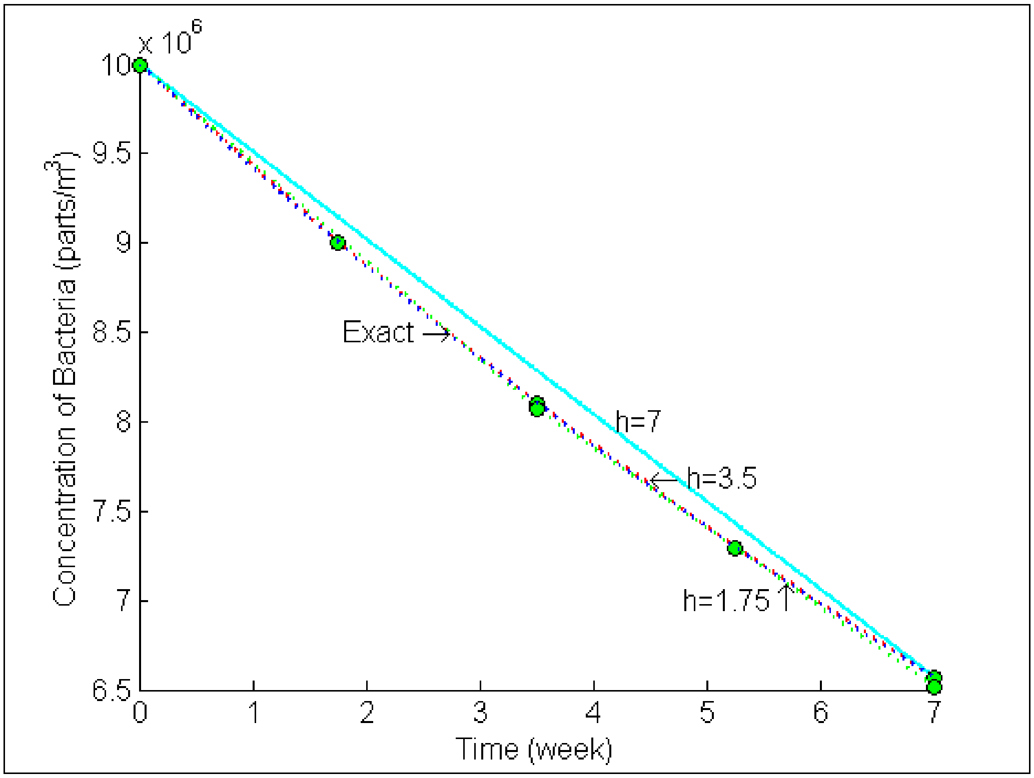

Figure 1 Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.

Table 1 and Figure 2 shows the effect of step size on the value of the calculated concentration of bacteria at $t = 7$ weeks.

Table 1 Value of concentration of bacteria at 7 weeks for different step sizes.

Step size, h	$C(7)$	E_t	$\lvert\in_t\rvert\,\%$
7	6.5715×10^6	-1017.2	0.015481
3.5	6.5705×10^6	-53.301	8.1121×10^{-4}
1.75	6.5705×10^6	-3.0512	4.6438×10^{-5}
0.875	6.5705×10^6	-0.18252	2.7779×10^{-6}
0.4375	6.5705×10^6	-0.011161	1.6986×10^{-7}

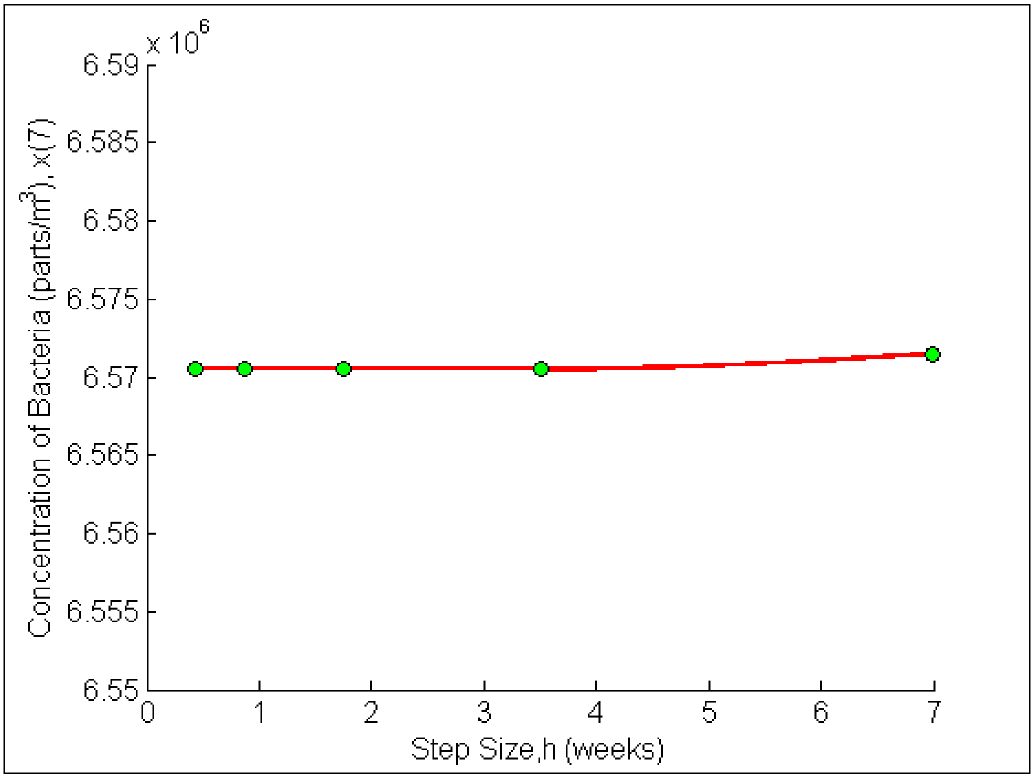

Figure 2 Effect of step size in Runge-Kutta 4^{th} order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1^{st} order method), Heun's method (Runge-Kutta 2^{nd} order method) and the Runge-Kutta 4^{th} order method.

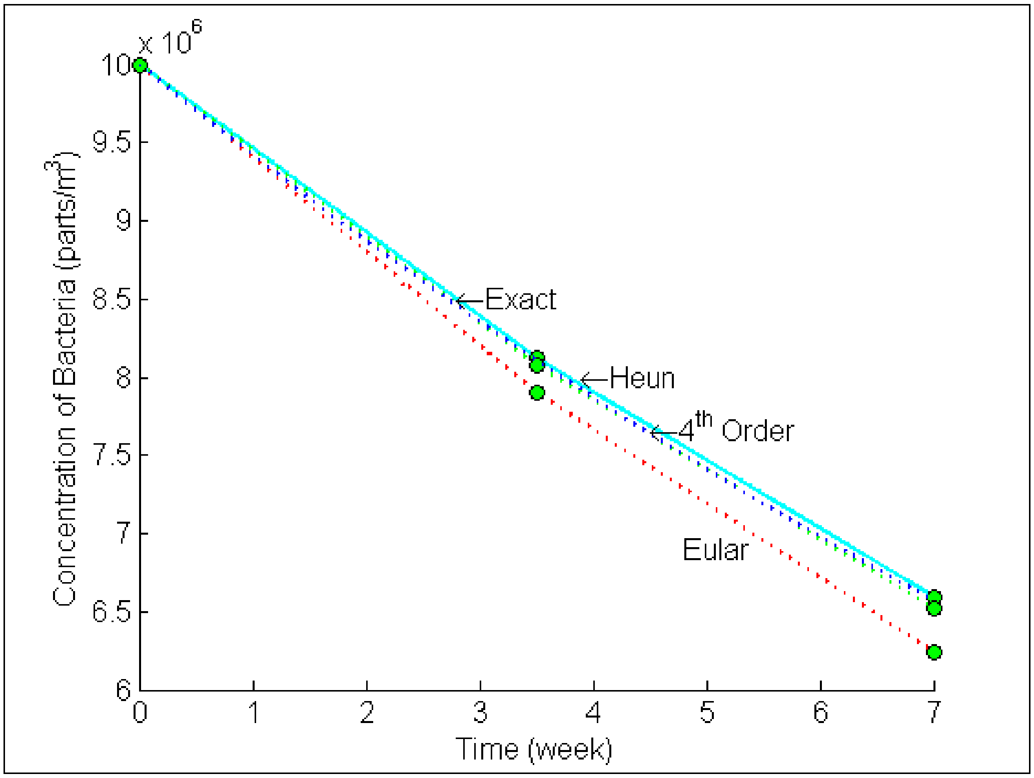

Figure 3 Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Runge-Kutta 4th order method
Summary	Textbook notes on the Runge-Kutta 4th order method for solving ordinary differential equations.
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