Runge 2nd Order Method

Computer Engineering Majors

Authors: Autar Kaw, Charlie Barker

http://numericalmethods.eng.usf.edu

Transforming Numerical Methods Education for STEM Undergraduates

Runge-Kutta 2^{nd} Order Method

http://numericalmethods.eng.usf.edu

Runge-Kutta 2nd Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 2nd order method is given by

$$y_{i+1} = y_i + \left(a_1 k_1 + a_2 k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + p_1 h, y_i + q_{11} k_1 h\right)$$

Heun's Method

<u>Heun's method</u>

Here a_2=1/2 is chosen

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

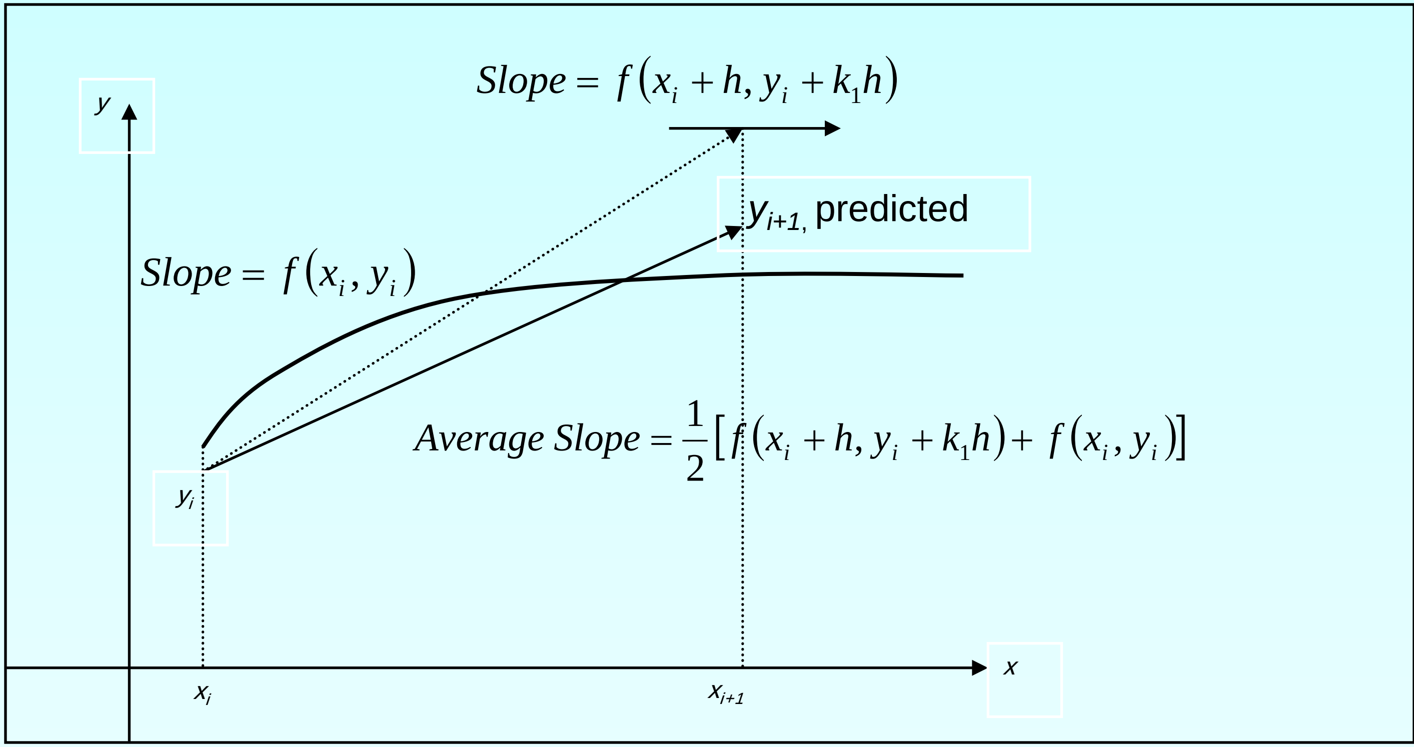

Figure 1 Runge-Kutta 2nd order method (Heun's method)

Midpoint Method

Here $a_2 = 1$ **is chosen, giving**

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

Ralston's Method

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, \; y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, \; y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A rectifier-based power supply requires a capacitor to temporarily store power when the rectified waveform from the AC source drops below the target voltage. To properly size this capacitor a first-order ordinary differential equation must be solved. For a particular power supply, with a capacitor of 150 μF, the ordinary differential equation to be solved is

$$\frac{dv(t)}{dt}=\frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v(t)}{0.04},0\right)\right\}$$

Find voltage across the capacitor at t= 0.00004s. Use step size h=0.00002

$$\frac{dv}{dt}=\frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v}{0.04},0\right)\right\}$$

$$f(t,v)=\frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v}{0.04},0\right)\right\}$$

$$v_{i+1}=v_i+\left(\frac{1}{2}k_1+\frac{1}{2}k_2\right)h$$

Solution

Step 1: $i = 0, \quad t_0 = 0, \quad v_0 = v(0) = 0$

$$k_1 = f\left(t_0, v_o\right) = f\left(0,0\right) = \frac{1}{150\times10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0))\right| - 2 - (0)}{0.04}, 0\right)\right\} = 2.6660\times10^{6}$$

$$k_2 = f\left(t_0 + h, v_0 + k_1 h\right) = f\left(0 + 0.00002, 0 + \left(2.6660\times10^{6}\right)0.00002\right) = f\left(0.00002,\ 53.32\right)$$

$$= \frac{1}{150\times10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0.00002))\right| - 2 - (53.32)}{0.04}, 0\right)\right\} = -666.67$$

$$v_1 = v_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h = 0 + \left(\frac{1}{2}\left(2.6660\times10^{6}\right) + \frac{1}{2}\left(-666.67\right)\right)0.00002$$

$$= 0 + \left(1.3327\times10^{6}\right)0.00002 = 26.653\text{V}$$

Solution Cont

Step 2: $i=1,\quad t_1 = t_0 + h = 0 + 0.00002 = 0.00002 \quad v_1 = 26.653\,\text{V}$

$$k_1 = f(t_1, v_1) = f(0.00002,\ 26.653)$$

$$= \frac{1}{150\times10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0.00002))| - 2 - (26.653)}{0.04}, 0\right)\right\} = -666.67$$

$$k_2 = f(t_1 + h, v_1 + k_1 h) = f(0.00002 + 0.00002,\ 26.653 + (-666.67)0.00002) = f(0.00004,\ 26.640)$$

$$= \frac{1}{150\times10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0.00004))| - 2 - (26.640)}{0.04}, 0\right)\right\} = -666.67$$

$$v_2 = v_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h = 26.653 + \left(\frac{1}{2}(-666.67) + \frac{1}{2}(-666.67)\right)0.00002$$

$$= 26.653 + (-666.67)0.00002 = 26.647\text{V}$$

Solution Continued

The solution to this nonlinear equation at t=0.00004 seconds is

$$v(0.00004) = 15.974V$$

Comparison with exact results

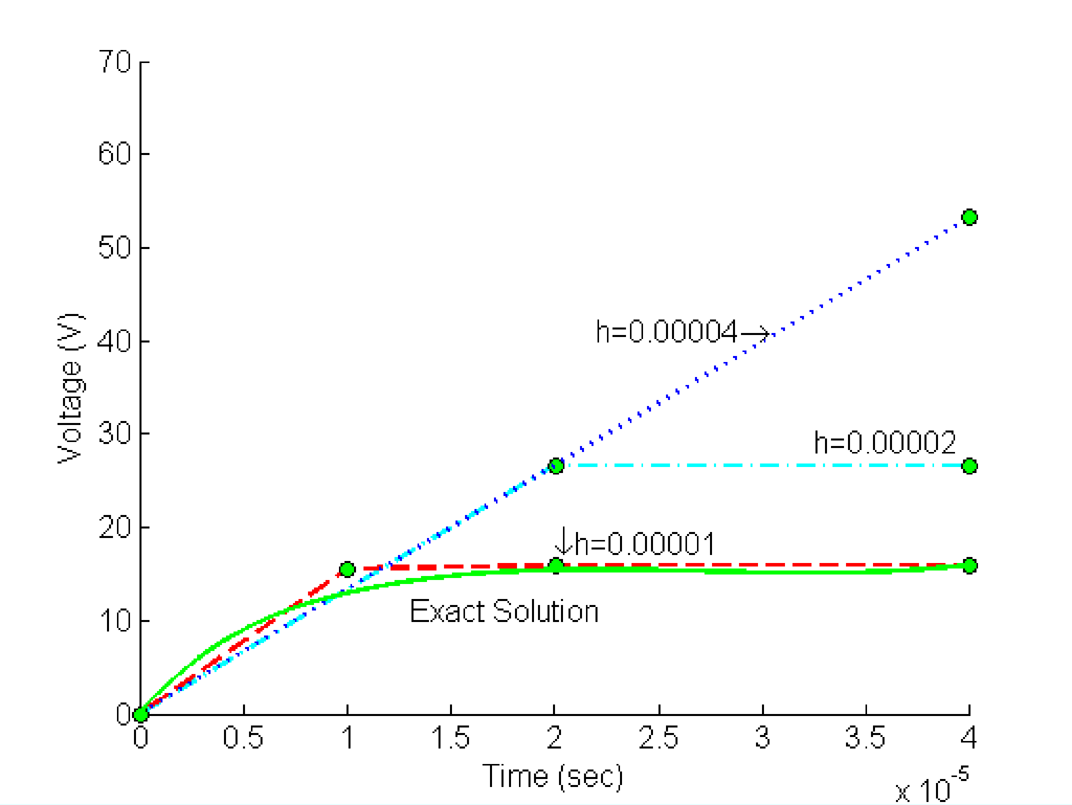

Figure 2. Heun's method results for different step sizes

Effect of step size

Table 1. Effect of step size for Heun's method

Step size, h	$v(0.00004)$	E_t	$\|\in_t\|\%$
0.00004	53.307	−37.333	233.71
0.00002	26.640	−10.666	65.771
0.00001	15.980	−0.0056605	0.035436
0.000005	15.918	0.055825	0.34947
0.0000025	15.970	0.0044682	0.027974

$v(0.00004) = 15.974\text{V}$ (exact)

Effects of step size on Heun's Method

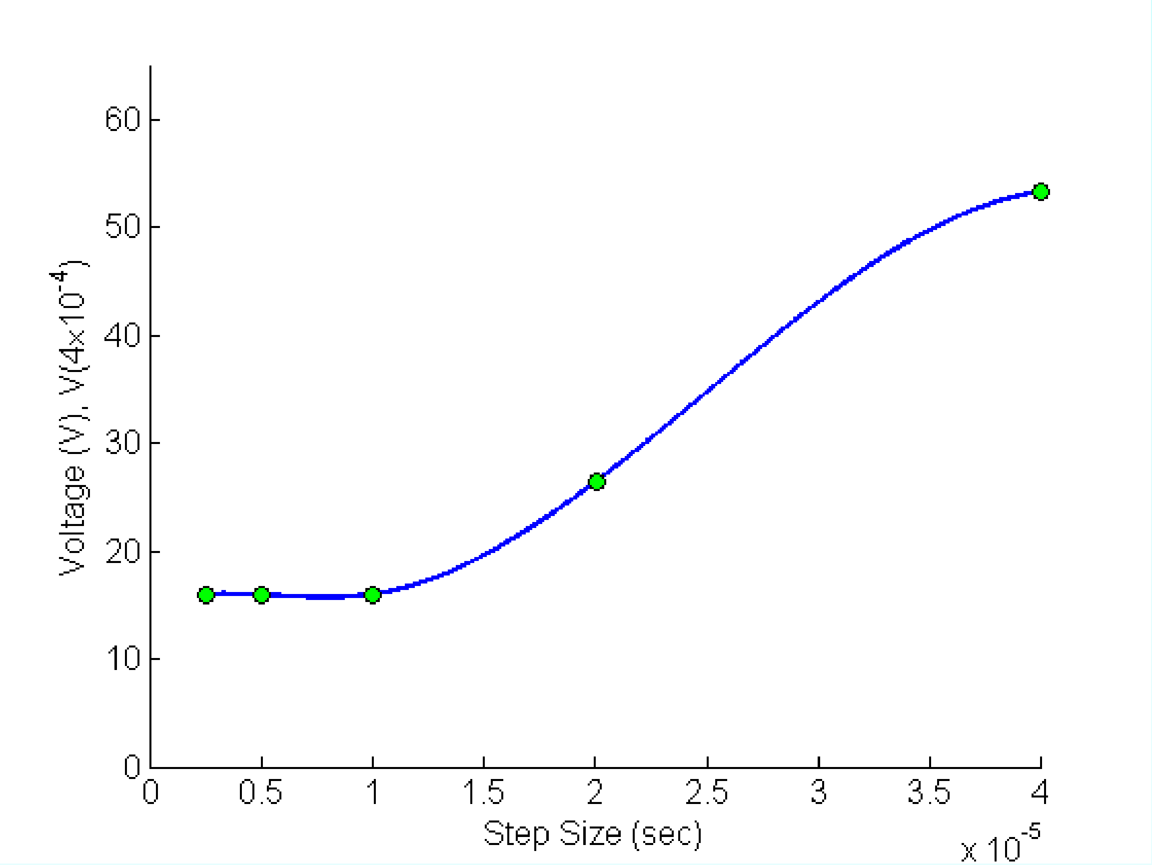

Figure 3. Effect of step size in Heun's method

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$v(0.00004)$			
	Euler	Heun	Midpoint	Ralston
0.00004	106.64	53.307	−0.026667	35.529
0.00002	53.307	26.640	−0.026667	17.751
0.00001	26.640	15.980	11.642	15.363
0.000005	15.996	15.918	15.917	15.917
0.0000025	15.993	15.970	15.968	15.968

$v(0.00004) = 15.974\text{V}$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\lvert \in_t \rvert \%$			
	Euler	Heun	Midpoint	Ralston
0.00004	567.59	233.71	100.17	122.47
0.00002	233.71	65.269	100.17	11.152
0.00001	66.771	0.031301	27.101	3.8009
0.000005	0.13146	0.35683	0.33187	0.33187
0.0000025	0.11268	0.037561	0.012523	0.012523

$v(0.00004) = 15.974V$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

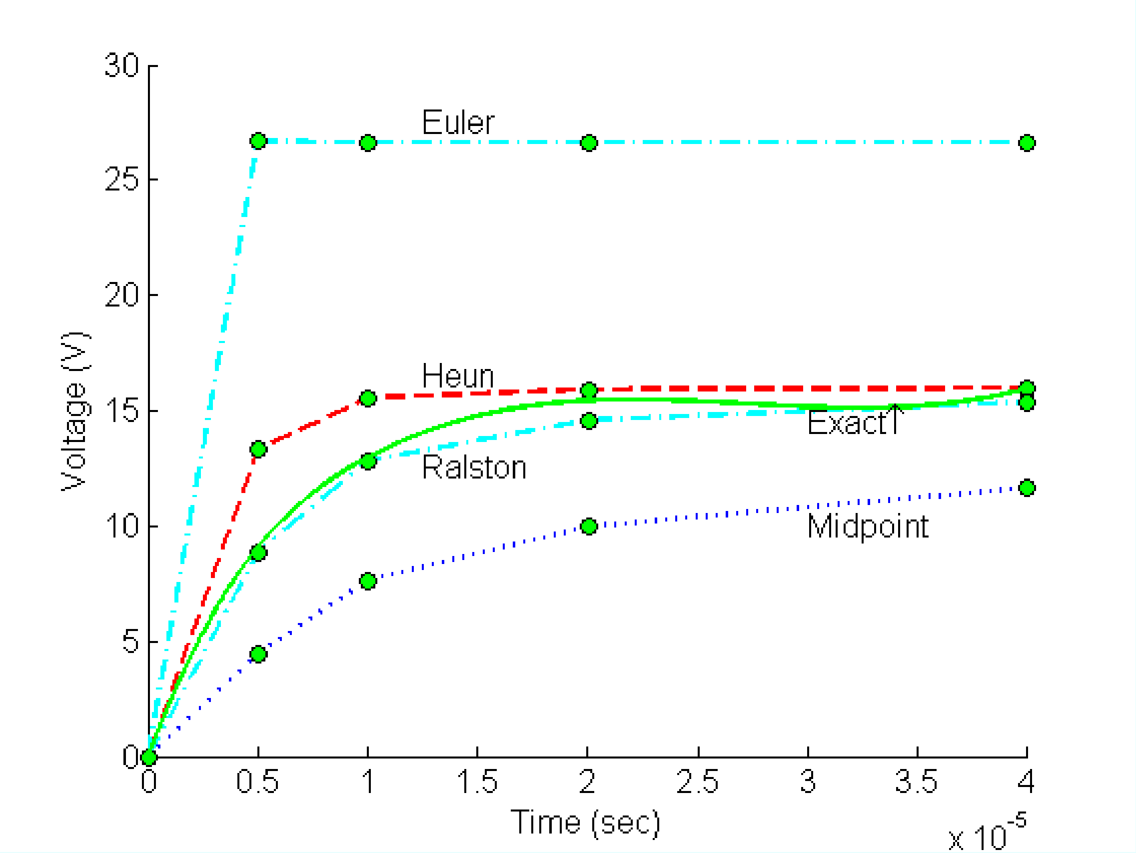

Figure 4. Comparison of Euler and Runge Kutta 2nd order methods with exact results.

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_2nd_method.html

THE END

http://numericalmethods.eng.usf.edu