Chapter 04.03
Binary Matrix Operations

After reading this chapter, you should be able to

1. *add, subtract, and multiply matrices, and*
2. *apply rules of binary operations on matrices.*

How do you add two matrices?

Two matrices $[A]$ and $[B]$ can be added only if they are the same size. The addition is then shown as

$$[C]=[A]+[B]$$

where

$$c_{ij}=a_{ij}+b_{ij}$$

Example 1

Add the following two matrices.

$$[A]=\begin{bmatrix}5 & 2 & 3\\ 1 & 2 & 7\end{bmatrix} \qquad [B]=\begin{bmatrix}6 & 7 & -2\\ 3 & 5 & 19\end{bmatrix}$$

Solution

$$\begin{aligned}[C]&=[A]+[B]\\ &=\begin{bmatrix}5 & 2 & 3\\ 1 & 2 & 7\end{bmatrix}+\begin{bmatrix}6 & 7 & -2\\ 3 & 5 & 19\end{bmatrix}\\ &=\begin{bmatrix}5+6 & 2+7 & 3-2\\ 1+3 & 2+5 & 7+19\end{bmatrix}\\ &=\begin{bmatrix}11 & 9 & 1\\ 4 & 7 & 26\end{bmatrix}\end{aligned}$$

Example 2

Blowout r'us store has two store locations A and B, and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3 and 4. What are the total tire sales for the two locations by make and quarter?

Solution

$$[C] = [A] + [B]$$

$$= \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} + \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} (25+20) & (20+5) & (3+4) & (2+0) \\ (5+3) & (10+6) & (15+15) & (25+21) \\ (6+4) & (16+1) & (7+7) & (27+20) \end{bmatrix}$$

$$= \begin{bmatrix} 45 & 25 & 7 & 2 \\ 8 & 16 & 30 & 46 \\ 10 & 17 & 14 & 47 \end{bmatrix}$$

So if one wants to know the total number of Copper tires sold in quarter 4 at the two locations, we would look at Row 3 – Column 4 to give $c_{34} = 47$.

How do you subtract two matrices?

Two matrices $[A]$ and $[B]$ can be subtracted only if they are the same size. The subtraction is then given by

$$[D] = [A] - [B]$$

Where

$$d_{ij} = a_{ij} - b_{ij}$$

Example 3

Subtract matrix $[B]$ from matrix $[A]$.

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$

Solution

$$[D] = [A] - [B]$$
$$= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$
$$= \begin{bmatrix} (5-6) & (2-7) & (3-(-2)) \\ (1-3) & (2-5) & (7-19) \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -5 & 5 \\ -2 & -3 & -12 \end{bmatrix}$$

Example 4

Blowout r'us has two store locations A and B and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$
$$[B] = \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3, and 4. How many more tires did store A sell than store B of each brand in each quarter?

Solution

$$[D] = [A] - [B]$$
$$= \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} - \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$
$$= \begin{bmatrix} 25-20 & 20-5 & 3-4 & 2-0 \\ 5-3 & 10-6 & 15-15 & 25-21 \\ 6-4 & 16-1 & 7-7 & 27-20 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 15 & -1 & 2 \\ 2 & 4 & 0 & 4 \\ 2 & 15 & 0 & 7 \end{bmatrix}$$

So if you want to know how many more Copper tires were sold in quarter 4 in store A than store B, $d_{34} = 7$. Note that $d_{13} = -1$ implies that store A sold 1 less Michigan tire than store B in quarter 3.

How do I multiply two matrices?

Two matrices $[A]$ and $[B]$ can be multiplied only if the number of columns of $[A]$ is equal to the number of rows of $[B]$ to give

$$[C]_{m\times n}=[A]_{m\times p}[B]_{p\times n}$$

If $[A]$ is a $m\times p$ matrix and $[B]$ is a $p\times n$ matrix, the resulting matrix $[C]$ is a $m\times n$ matrix.

So how does one calculate the elements of $[C]$ matrix?

$$\begin{aligned}c_{ij}&=\sum_{k=1}^{p}a_{ik}b_{kj}\\&=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots\ldots+a_{ip}b_{pj}\end{aligned}$$

for each $i=1,\ 2,\ldots\ldots,m$ and $j=1,\ 2,\ldots\ldots,n$.

To put it in simpler terms, the i^{th} row and j^{th} column of the $[C]$ matrix in $[C]=[A][B]$ is calculated by multiplying the i^{th} row of $[A]$ by the j^{th} column of $[B]$, that is,

$$\begin{aligned}c_{ij}&=\begin{bmatrix}a_{i1}\ a_{i2}\ldots\ldots a_{ip}\end{bmatrix}\begin{bmatrix}b_{1j}\\b_{2j}\\\vdots\\\vdots\\b_{pj}\end{bmatrix}\\&=a_{i1}\,b_{1j}+\ a_{i2}\,b_{2j}+\ldots\ldots.+a_{ip}\,b_{pj}.\\&=\sum_{k=1}^{p}a_{ik}b_{kj}\end{aligned}$$

Example 5

Given

$$[A]=\begin{bmatrix}5&2&3\\1&2&7\end{bmatrix}$$

$$[B]=\begin{bmatrix}3&-2\\5&-8\\9&-10\end{bmatrix}$$

Find

$$[C]=[A][B]$$

Solution

c_{12} can be found by multiplying the first row of $[A]$ by the second column of $[B]$,

$$c_{12}=\begin{bmatrix}5&2&3\end{bmatrix}\begin{bmatrix}-2\\-8\\-10\end{bmatrix}$$

$$= (5)(-2) + (2)(-8) + (3)(-10)$$
$$= -56$$

Similarly, one can find the other elements of $[C]$ to give

$$[C] = \begin{bmatrix} 52 & -56 \\ 76 & -88 \end{bmatrix}$$

Example 6

Blowout r'us store location A and the sales of tires are given by make (in rows) and quarters (in columns) as shown below

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3, and 4. Find the per quarter sales of store A if the following are the prices of each tire.

Tirestone = $33.25
Michigan = $40.19
Copper = $25.03

Solution

The answer is given by multiplying the price matrix by the quantity of sales of store A. The price matrix is $[33.25 \quad 40.19 \quad 25.03]$, so the per quarter sales of store A would be given by

$$[C] = [33.25 \quad 40.19 \quad 25.03] \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^{3} a_{ik} b_{kj}$$

$$c_{11} = \sum_{k=1}^{3} a_{1k} b_{k1}$$
$$= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$
$$= -(33.25)(25) + (40.19)(5) + (25.03)(6)$$
$$= \$1747.06$$

Similarly

$$c_{12} = \$1467.38$$
$$c_{13} = \$877.81$$
$$c_{14} = \$1747.06$$

Therefore, each quarter sales of store A in dollars is given by the four columns of the row vector

$$[C] = [1182.38 \quad 1467.38 \quad 877.81 \quad 1747.06]$$

Remember since we are multiplying a 1×3 matrix by a 3×4 matrix, the resulting matrix is a 1×4 matrix.

What is the scalar product of a constant and a matrix?

If $[A]$ is a $n\times n$ matrix and k is a real number, then the scalar product of k and $[A]$ is another $n\times n$ matrix $[B]$, where $b_{ij} = k\,a_{ij}$.

Example 7

Let

$$[A]=\begin{bmatrix}2.1 & 3 & 2\\ 5 & 1 & 6\end{bmatrix}$$

Find $2[A]$

Solution

$$2[A]=2\begin{bmatrix}2.1 & 3 & 2\\ 5 & 1 & 6\end{bmatrix}$$

$$=\begin{bmatrix}2\times 2.1 & 2\times 3 & 2\times 2\\ 2\times 5 & 2\times 1 & 2\times 6\end{bmatrix}$$

$$=\begin{bmatrix}4.2 & 6 & 4\\ 10 & 2 & 12\end{bmatrix}$$

What is a linear combination of matrices?

If $[A_1],[A_2],.....,[A_p]$ are matrices of the same size and $k_1,k_2,.....,k_p$ are scalars, then

$$k_1[A_1]+k_2[A_2]+........+k_p[A_p]$$

is called a linear combination of $[\mathrm{A}_1],[A_2],.....,[A_p]$

Example 8

If $[A_1]=\begin{bmatrix}5 & 6 & 2\\ 3 & 2 & 1\end{bmatrix}, [A_2]=\begin{bmatrix}2.1 & 3 & 2\\ 5 & 1 & 6\end{bmatrix}, [A_3]=\begin{bmatrix}0 & 2.2 & 2\\ 3 & 3.5 & 6\end{bmatrix}$

then find

$$[A_1]+2[A_2]-0.5[A_3]$$

Solution

$$[A_1]+2[A_2]-0.5[A_3]$$

$$=\begin{bmatrix}5 & 6 & 2\\ 3 & 2 & 1\end{bmatrix}+2\begin{bmatrix}2.1 & 3 & 2\\ 5 & 1 & 6\end{bmatrix}-0.5\begin{bmatrix}0 & 2.2 & 2\\ 3 & 3.5 & 6\end{bmatrix}$$

$$=\begin{bmatrix}5 & 6 & 2\\ 3 & 2 & 1\end{bmatrix}+\begin{bmatrix}4.2 & 6 & 4\\ 10 & 2 & 12\end{bmatrix}-\begin{bmatrix}0 & 1.1 & 1\\ 1.5 & 1.75 & 3\end{bmatrix}$$

$$=\begin{bmatrix}9.2 & 10.9 & 5\\ 11.5 & 2.25 & 10\end{bmatrix}$$

What are some of the rules of binary matrix operations?

Commutative law of addition
If $[A]$ and $[B]$ are $m \times n$ matrices, then

$$[A]+[B]=[B]+[A]]$$

Associative law of addition

If [A], [B] and [C] are all $m \times n$ matrices, then

$$[A]+([B]+[C])=([A]+[B])+[C]$$

Associative law of multiplication

If $[A]$, $[B]$ and $[C]$ are $m \times n, n \times p$ and $p \times r$ size matrices, respectively, then

$$[A]([B][C])=([A][B])[C]$$

and the resulting matrix size on both sides of the equation is $m \times r$.

Distributive law

If $[A]$ and $[B]$ are $m \times n$ size matrices, and $[C]$ and $[D]$ are $n \times p$ size matrices

$$[A]([C]+[D])=[A][C]+[A][D]$$

$$([A]+[B])[C]=[A][C]+[B][C]$$

and the resulting matrix size on both sides of the equation is $m \times p$.

Example 9

Illustrate the associative law of multiplication of matrices using

$$[A]=\begin{bmatrix}1 & 2\\ 3 & 5\\ 0 & 2\end{bmatrix},\quad [B]=\begin{bmatrix}2 & 5\\ 9 & 6\end{bmatrix},\quad [C]=\begin{bmatrix}2 & 1\\ 3 & 5\end{bmatrix}$$

Solution

$$[B][C]=$$

$$=\begin{bmatrix}2 & 5\\ 9 & 6\end{bmatrix}\begin{bmatrix}2 & 1\\ 3 & 5\end{bmatrix}$$

$$=\begin{bmatrix}19 & 27\\ 36 & 39\end{bmatrix}$$

$$[A]([B][C])=\begin{bmatrix}1 & 2\\ 3 & 5\\ 0 & 2\end{bmatrix}\begin{bmatrix}19 & 27\\ 36 & 39\end{bmatrix}$$

$$=\begin{bmatrix}91 & 105\\ 237 & 276\\ 72 & 78\end{bmatrix}$$

$$[A][B]=\begin{bmatrix}1 & 2\\ 3 & 5\\ 0 & 2\end{bmatrix}\begin{bmatrix}2 & 5\\ 9 & 6\end{bmatrix}$$

$$=\begin{bmatrix}20 & 17\\ 51 & 45\\ 18 & 12\end{bmatrix}$$

$$([A][B])[C]=\begin{bmatrix}20 & 17\\ 51 & 45\\ 18 & 12\end{bmatrix}\begin{bmatrix}2 & 1\\ 3 & 5\end{bmatrix}$$

$$=\begin{bmatrix}91 & 105\\ 237 & 276\\ 72 & 78\end{bmatrix}$$

The above illustrates the associative law of multiplication of matrices.

Is [A][B] = [B][A]?

If $[A]\,[B]$ exists, number of columns of $[A]$ has to be same as the number of rows of $[B]$ and if $[B][A]$ exists, number of columns of $[B]$ has to be same as the number of rows of $[A]$. Now for $[A][B]=[B][A]$, the resulting matrix from $[A][B]$ and $[B][A]$ has to be of the same size. This is only possible if $[A]$ and $[B]$ are square and are of the same size. Even then in general $[A][B]\neq[B][A]$

Example 10

Determine if

$$[A][B]=[B][A]$$

for the following matrices

$$[A]=\begin{bmatrix}6 & 3\\ 2 & 5\end{bmatrix},\quad [B]=\begin{bmatrix}-3 & 2\\ 1 & 5\end{bmatrix}$$

Solution

$$[A][B]=\begin{bmatrix}6 & 3\\ 2 & 5\end{bmatrix}\begin{bmatrix}-3 & 2\\ 1 & 5\end{bmatrix}$$

$$=\begin{bmatrix}-15 & 27\\ -1 & 29\end{bmatrix}$$

$$[B][A]=\begin{bmatrix}-3 & 2\\ 1 & 5\end{bmatrix}\begin{bmatrix}6 & 3\\ 2 & 5\end{bmatrix}$$

$$=\begin{bmatrix}-14 & 1\\ 16 & 28\end{bmatrix}$$

$$[A][B] \neq [B][A]$$

Key Terms:

Addition of matrices
Subtraction of matrices
Multiplication of matrices
Scalar Product of matrices
Linear Combination of Matrices
Rules of Binary Matrix Operation