Chapter 04.04
Unary Matrix Operations

After reading this chapter, you should be able to:

1. *know what unary operations means,*
2. *find the transpose of a square matrix and it's relationship to symmetric matrices,*
3. *find the trace of a matrix, and*
4. *find the determinant of a matrix by the cofactor method.*

What is the transpose of a matrix?

Let $[A]$ be a $m \times n$ matrix. Then $[B]$ is the transpose of the $[A]$ if $b_{ji} = a_{ij}$ for all i and j. That is, the i^{th} row and the j^{th} column element of $[A]$ is the j^{th} row and i^{th} column element of $[B]$. Note, $[B]$ would be a $n \times m$ matrix. The transpose of $[A]$ is denoted by $[A]^{\mathrm{T}}$.

Example 1

Find the transpose of

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

Solution

The transpose of $[A]$ is

$$[A]^{\mathrm{T}} = \begin{bmatrix} 25 & 5 & 6 \\ 20 & 10 & 16 \\ 3 & 15 & 7 \\ 2 & 25 & 27 \end{bmatrix}$$

Note, the transpose of a row vector is a column vector and the transpose of a column vector is a row vector.

Also, note that the transpose of a transpose of a matrix is the matrix itself, that is, $\left([A]^{\mathrm{T}}\right)^{\mathrm{T}} = [A]$. Also, $(A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}; (cA)^{\mathrm{T}} = cA^{\mathrm{T}}$.

What is a symmetric matrix?

A square matrix $[A]$ with real elements where $a_{ij} = a_{ji}$ for $i = 1,2,...,n$ and $j = 1,2,...,n$ is called a symmetric matrix. This is same as, if $[A] = [A]^{\mathrm{T}}$, then $[A]^{\mathrm{T}}$ is a symmetric matrix.

Example 2

Give an example of a symmetric matrix.

Solution

$$[A] = \begin{bmatrix} 21.2 & 3.2 & 6 \\ 3.2 & 21.5 & 8 \\ 6 & 8 & 9.3 \end{bmatrix}$$

is a symmetric matrix as $a_{12} = a_{21} = 3.2$, $a_{13} = a_{31} = 6$ and $a_{23} = a_{32} = 8$.

What is a skew-symmetric matrix?

A $n \times n$ matrix is skew symmetric if $a_{ij} = -a_{ji}$ for $i = 1,...,n$ and $j = 1,...,n$. This is same as

$$[A] = -[A]^{\mathrm{T}}.$$

Example 3

Give an example of a skew-symmetric matrix.

Solution

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -5 \\ -2 & 5 & 0 \end{bmatrix}$$

is skew-symmetric as

$a_{12} = -a_{21} = 1$; $a_{13} = -a_{31} = 2$; $a_{23} = -a_{32} = -5$. Since $a_{ii} = -a_{ii}$ only if $a_{ii} = 0$, all the diagonal elements of a skew-symmetric matrix have to be zero.

What is the trace of a matrix?

The trace of a $n \times n$ matrix $[A]$ is the sum of the diagonal entries of $[A]$, that is,

$$\mathrm{tr}[A] = \sum_{i=1}^{n} a_{ii}$$

Example 4

Find the trace of

$$[A] = \begin{bmatrix} 15 & 6 & 7 \\ 2 & -4 & 2 \\ 3 & 2 & 6 \end{bmatrix}$$

Solution

$$\begin{aligned}\operatorname{tr}[A] &= \sum_{i=1}^{3} a_{ii} \\ &= (15) + (-4) + (6) \\ &= 17\end{aligned}$$

Example 5

The sales of tires are given by make (rows) and quarters (columns) for Blowout r'us store location A, as shown below.

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

where the rows represent the sale of Tirestone, Michigan and Copper tires, and the columns represent the quarter number 1, 2, 3, 4.
Find the total yearly revenue of store A if the prices of tires vary by quarters as follows.

$$[B] = \begin{bmatrix} 33.25 & 30.01 & 35.02 & 30.05 \\ 40.19 & 38.02 & 41.03 & 38.23 \\ 25.03 & 22.02 & 27.03 & 22.95 \end{bmatrix}$$

where the rows represent the cost of each tire made by Tirestone, Michigan and Copper, and the columns represent the quarter numbers.

Solution

To find the total tire sales of store A for the whole year, we need to find the sales of each brand of tire for the whole year and then add to find the total sales. To do so, we need to rewrite the price matrix so that the quarters are in rows and the brand names are in the columns, that is, find the transpose of $[B]$.

$$\begin{aligned}[C] &= [B]^{\mathrm{T}} \\ &= \begin{bmatrix} 33.25 & 30.01 & 35.02 & 30.05 \\ 40.19 & 38.02 & 41.03 & 38.23 \\ 25.03 & 22.02 & 27.03 & 22.95 \end{bmatrix}^{\mathrm{T}}\end{aligned}$$

$$[C] = \begin{bmatrix} 33.25 & 40.19 & 25.03 \\ 30.01 & 38.02 & 22.02 \\ 35.02 & 41.03 & 27.03 \\ 30.05 & 38.23 & 22.95 \end{bmatrix}$$

Recognize now that if we find $[A][C]$, we get

$$[D] = [A][C]$$

$$= \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} \begin{bmatrix} 33.25 & 40.19 & 25.03 \\ 30.01 & 38.02 & 22.02 \\ 35.02 & 41.03 & 27.03 \\ 30.05 & 38.23 & 22.95 \end{bmatrix}$$

$$= \begin{bmatrix} 1597 & 1965 & 1193 \\ 1743 & 2152 & 1325 \\ 1736 & 2169 & 1311 \end{bmatrix}$$

The diagonal elements give the sales of each brand of tire for the whole year, that is

$$d_{11} = \$1597 \quad \text{(Tirestone sales)}$$
$$d_{22} = \$2152 \quad \text{(Michigan sales)}$$
$$d_{33} = \$1311 \quad \text{(Cooper sales)}$$

The total yearly sales of all three brands of tires are

$$\sum_{i=1}^{3} d_{ii} = 1597 + 2152 + 1311$$
$$= \$5060$$

and this is the trace of the matrix $[A]$.

Define the determinant of a matrix.

The determinant of a square matrix is a single unique real number corresponding to a matrix. For a matrix $[A]$, determinant is denoted by $|A|$ or $\det(A)$. So do not use $[A]$ and $|A|$ interchangeably.

For a 2×2 matrix,

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

How does one calculate the determinant of any square matrix?

Let $[A]$ be $n \times n$ matrix. The minor of entry a_{ij} is denoted by M_{ij} and is defined as the determinant of the $(n-1 \times (n-1)$ submatrix of $[A]$, where the submatrix is obtained by deleting the i^{th} row and j^{th} column of the matrix $[A]$. The determinant is then given by

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \ \textit{for} \text{ any } i = 1, 2, \cdots, n$$

or

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \ \textit{for any } j = 1, 2, \cdots, n$$

coupled with that $\det(A) = a_{11}$ for a 1×1 matrix $[A]$, as we can always reduce the determinant of a matrix to determinants of 1×1 matrices. The number $(-1)^{i+j} M_{ij}$ is called the cofactor of a_{ij} and is denoted by c_{ij}. The above equation for the determinant can then be written as

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} \quad for\ any\ i = 1, 2, \cdots, n$$

or

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} \quad for\ any\ j = 1, 2, \cdots, n$$

The only reason why determinants are not generally calculated using this method is that it becomes computationally intensive. For a $n \times n$ matrix, it requires arithmetic operations proportional to n!.

Example 6

Find the determinant of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

Method 1:

$$\det(A) = \sum_{j=1}^{3} (-1)^{i+j} a_{ij} M_{ij} \text{ for any } i = 1, 2, 3$$

Let $i = 1$ in the formula

$$\det(A) = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} M_{1j}$$

$$= (-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} + (-1)^{1+3} a_{13} M_{13}$$

$$= a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$

$$M_{11} = \begin{vmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 8 & 1 \\ 12 & 1 \end{vmatrix}$$

$$= -4$$

$$M_{12} = \begin{vmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 64 & 1 \\ 144 & 1 \end{vmatrix}$$
$$= -80$$
$$M_{13} = \begin{vmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} 64 & 8 \\ 144 & 12 \end{vmatrix}$$
$$= -384$$
$$\det(A) = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$
$$= 25(-4) - 5(-80) + 1(-384)$$
$$= -100 + 400 - 384$$
$$= -84$$

Also for $i = 1$,
$$\det(A) = \sum_{j=1}^{3} a_{1j}C_{1j}$$
$$C_{11} = (-1)^{1+1} M_{11}$$
$$= M_{11}$$
$$= -4$$
$$C_{12} = (-1)^{1+2} M_{12}$$
$$= -M_{12}$$
$$= 80$$
$$C_{13} = (-1)^{1+3} M_{13}$$
$$= M_{13}$$
$$= -384$$
$$\det(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$
$$= (25)(-4) + (5)(80) + (1)(-384)$$
$$= -100 + 400 - 384$$
$$= -84$$

Method 2:

$$\det(A) = \sum_{i=1}^{3} (-1)^{i+j} a_{ij} M_{ij} \quad \text{for any } j = 1, 2, 3$$

Let $j = 2$ in the formula
$$\det(A) = \sum_{i=1}^{3} (-1)^{i+2} a_{i2} M_{i2}$$
$$= (-1)^{1+2} a_{12}M_{12} + (-1)^{2+2} a_{22}M_{22} + (-1)^{3+2} a_{32}M_{32}$$
$$= -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32}$$

$$M_{12} = \begin{vmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} 64 & 1 \\ 144 & 1 \end{vmatrix}$$
$$= -80$$
$$M_{22} = \begin{vmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} 25 & 1 \\ 144 & 1 \end{vmatrix}$$
$$= -119$$
$$M_{32} = \begin{vmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} 25 & 1 \\ 64 & 1 \end{vmatrix}$$
$$= -39$$
$$\begin{aligned} \det(A) &= -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32} \\ &= -5(-80) + 8(-119) - 12(-39) \\ &= 400 - 952 + 468 \\ &= -84 \end{aligned}$$

In terms of cofactors for $j = 2$,

$$\det(A) = \sum_{i=1}^{3} a_{i2}C_{i2}$$
$$\begin{aligned} C_{12} &= (-1)^{1+2} M_{12} \\ &= -M_{12} \\ &= 80 \end{aligned}$$
$$\begin{aligned} C_{22} &= (-1)^{2+2} M_{22} \\ &= M_{22} \\ &= -119 \end{aligned}$$
$$\begin{aligned} C_{32} &= (-1)^{3+2} M_{32} \\ &= -M_{32} \\ &= 39 \end{aligned}$$
$$\begin{aligned} \det(A) &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= (5)(80) + (8)(-119) + (12)(39) \\ &= 400 - 952 + 468 \end{aligned}$$

$$= -84$$

Is there a relationship between det(*AB*), and det(*A*) and det(*B*)?

Yes, if $[A]$ and $[B]$ are square matrices of same size, then

$$\det(AB) = \det(A)\det(B)$$

Are there some other theorems that are important in finding the determinant of a square matrix?

Theorem 1: If a row or a column in a $n \times n$ matrix $[A]$ is zero, then $\det(A) = 0$.

Theorem 2: Let $[A]$ be a $n \times n$ matrix. If a row is proportional to another row, then $\det(A) = 0$.

Theorem 3: Let $[A]$ be a $n \times n$ matrix. If a column is proportional to another column, then $\det(A) = 0$.

Theorem 4: Let $[A]$ be a $n \times n$ matrix. If a column or row is multiplied by k to result in matrix k, then $\det(B) = k\det(A)$.

Theorem 5: Let $[A]$ be a $n \times n$ upper or lower triangular matrix, then $\det(B) = \underset{i=1}{\overset{n}{\pi}} a_{ii}$.

Example 7

What is the determinant of

$$[A] = \begin{bmatrix} 0 & 2 & 6 & 3 \\ 0 & 3 & 7 & 4 \\ 0 & 4 & 9 & 5 \\ 0 & 5 & 2 & 1 \end{bmatrix}$$

Solution

Since one of the columns (first column in the above example) of $[A]$ is a zero, $\det(A) = 0$.

Example 8

What is the determinant of

$$[A] = \begin{bmatrix} 2 & 1 & 6 & 4 \\ 3 & 2 & 7 & 6 \\ 5 & 4 & 2 & 10 \\ 9 & 5 & 3 & 18 \end{bmatrix}$$

Solution

$\det(A)$ is zero because the fourth column

$$\begin{bmatrix} 4 \\ 6 \\ 10 \\ 18 \end{bmatrix}$$

is 2 times the first column

$$\begin{bmatrix} 2 \\ 3 \\ 5 \\ 9 \end{bmatrix}$$

Example 9

If the determinant of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

is -84, then what is the determinant of

$$[B] = \begin{bmatrix} 25 & 10.5 & 1 \\ 64 & 16.8 & 1 \\ 144 & 25.2 & 1 \end{bmatrix}$$

Solution

Since the second column of $[B]$ is 2.1 times the second column of $[A]$

$$\begin{aligned} \det(B) &= 2.1\det(A) \\ &= (2.1)(-84) \\ &= -176.4 \end{aligned}$$

Example 10

Given the determinant of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

is -84, what is the determinant of

$$[B] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

Since $[B]$ is simply obtained by subtracting the second row of $[A]$ by 2.56 times the first row of $[A]$,

$$\begin{aligned}\det(\mathrm{B}) &= \det(\mathrm{A}) \\ &= -84\end{aligned}$$

Example 11

What is the determinant of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Solution

Since $[A]$ is an upper triangular matrix

$$\begin{aligned}\det(A) &= \prod_{i=1}^{3} a_{ii} \\ &= (a_{11})(a_{22})(a_{33}) \\ &= (25)(-4.8)(0.7) \\ &= -84\end{aligned}$$

Key Terms

Transpose
Symmetric Matrix
Skew-Symmetric Matrix
Trace of Matrix
Determinant