Chapter 04.02
Vectors

After reading this chapter, you should be able to:

1. *define a vector,*
2. *add and subtract vectors,*
3. *find linear combinations of vectors and their relationship to a set of equations,*
4. *explain what it means to have a linearly independent set of vectors, and*
5. *find the rank of a set of vectors.*

What is a vector?

A vector is a collection of numbers in a definite order. If it is a collection of n numbers, it is called a n-dimensional vector. So the vector $\vec{A}$ given by

$$\vec{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

is a n-dimensional column vector with n components, $a_1, a_2, \ldots\ldots, a_n$. The above is a column vector. A row vector $[B]$ is of the form $\vec{B} = [b_1, b_2, \ldots., b_n]$ where $\vec{B}$ is a n-dimensional row vector with n components $b_1, b_2, \ldots., b_n$.

Example 1

Give an example of a 3-dimensional column vector.

Solution

Assume a point in space is given by its (x, y, z) coordinates. Then if the value of $x = 3, y = 2, z = 5$, the column vector corresponding to the location of the points is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}.$$

When are two vectors equal?

Two vectors $\vec{A}$ and $\vec{B}$ are equal if they are of the same dimension and if their corresponding components are equal.
Given

$$\vec{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

and

$$\vec{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

then $\vec{A} = \vec{B}$ if $a_i = b_i$, $i = 1,2,......,n$.

Example 2

What are the values of the unknown components in $\vec{B}$ if

$$\vec{A} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

and

$$\vec{B} = \begin{bmatrix} b_1 \\ 3 \\ 4 \\ b_4 \end{bmatrix}$$

and $\vec{A} = \vec{B}$.
Solution

$$b_1 = 2, b_4 = 1$$

How do you add two vectors?

Two vectors can be added only if they are of the same dimension and the addition is given by

$$[A]+[B]=\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}+\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$$

$$=\begin{bmatrix}a_1+b_1\\a_2+b_2\\\vdots\\a_n+b_n\end{bmatrix}$$

Example 3

Add the two vectors

$$\vec{A}=\begin{bmatrix}2\\3\\4\\1\end{bmatrix}$$

and

$$\vec{B}=\begin{bmatrix}5\\-2\\3\\7\end{bmatrix}$$

Solution

$$\vec{A}+\vec{B}=\begin{bmatrix}2\\3\\4\\1\end{bmatrix}+\begin{bmatrix}5\\-2\\3\\7\end{bmatrix}$$

$$=\begin{bmatrix}2+5\\3-2\\4+3\\1+7\end{bmatrix}$$

$$=\begin{bmatrix}7\\1\\7\\8\end{bmatrix}$$

Example 4

A store sells three brands of tires: Tirestone, Michigan and Copper. In quarter 1, the sales are given by the column vector

$$\vec{A}_1 = \begin{bmatrix} 25 \\ 5 \\ 6 \end{bmatrix}$$

where the rows represent the three brands of tires sold – Tirestone, Michigan and Copper respectively. In quarter 2, the sales are given by

$$\vec{A}_2 = \begin{bmatrix} 20 \\ 10 \\ 6 \end{bmatrix}$$

What is the total sale of each brand of tire in the first half of the year?

Solution

The total sales would be given by

$$\begin{aligned} \vec{C} &= \vec{A}_1 + \vec{A}_2 \\ &= \begin{bmatrix} 25 \\ 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 20 \\ 10 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 25+20 \\ 5+10 \\ 6+6 \end{bmatrix} \\ &= \begin{bmatrix} 45 \\ 15 \\ 12 \end{bmatrix} \end{aligned}$$

So the number of Tirestone tires sold is 45, Michigan is 15 and Copper is 12 in the first half of the year.

What is a null vector?

A null vector is where all the components of the vector are zero.

Example 5

Give an example of a null vector or zero vector.

Solution

The vector

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is an example of a zero or null vector.

What is a unit vector?

A unit vector $\vec{U}$ is defined as

$$\vec{U} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

where

$$\sqrt{u_1^2 + u_2^2 + u_3^2 + \ldots + u_n^2} = 1$$

Example 6

Give examples of 3-dimensional unit column vectors.

Solution

Examples include

$$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{etc.}$$

How do you multiply a vector by a scalar?

If k is a scalar and $\vec{A}$ is a n-dimensional vector, then

$$k\vec{A} = k\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$= \begin{bmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{bmatrix}$$

Example 7

What is $2\vec{A}$ if

$$\vec{A} = \begin{bmatrix} 25 \\ 20 \\ 5 \end{bmatrix}$$

Solution

$$2\vec{A} = 2\begin{bmatrix} 25 \\ 20 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 25 \\ 2 \times 20 \\ 2 \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 50 \\ 40 \\ 10 \end{bmatrix}$$

Example 8

A store sells three brands of tires: Tirestone, Michigan and Copper. In quarter 1, the sales are given by the column vector

$$\vec{A} = \begin{bmatrix} 25 \\ 25 \\ 6 \end{bmatrix}$$

If the goal is to increase the sales of all tires by at least 25% in the next quarter, how many of each brand should be sold?

Solution

Since the goal is to increase the sales by 25%, one would multiply the $\vec{A}$ vector by 1.25,

$$\vec{B} = 1.25\begin{bmatrix} 25 \\ 25 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 31.25 \\ 31.25 \\ 7.5 \end{bmatrix}$$

Since the number of tires must be an integer, we can say that the goal of sales is

$$\vec{B} = \begin{bmatrix} 32 \\ 32 \\ 8 \end{bmatrix}$$

What do you mean by a linear combination of vectors?

Given

$$\vec{A}_1, \vec{A}_2, \ldots\ldots, \vec{A}_m$$

as m vectors of same dimension n, and if $k_1, k_2, \ldots, k_m$ are scalars, then

$$k_1\vec{A}_1 + k_2\vec{A}_2 + \ldots\ldots + k_m\vec{A}_m$$

is a linear combination of the m vectors.

Example 9

Find the linear combinations

a) $\vec{A} - \vec{B}$ and

b) $\vec{A} + \vec{B} - 3\vec{C}$

where

$$\vec{A} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \vec{B} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{C} = \begin{bmatrix} 10 \\ 1 \\ 2 \end{bmatrix}$$

Solution

a) $\vec{A} - \vec{B} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 2-1 \\ 3-1 \\ 6-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

b) $\vec{A} + \vec{B} - 3\vec{C} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 10 \\ 1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 2+1-30 \\ 3+1-3 \\ 6+2-6 \end{bmatrix}$$

$$= \begin{bmatrix} -27 \\ 1 \\ 2 \end{bmatrix}$$

What do you mean by vectors being linearly independent?

A set of vectors $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_m$ are considered to be linearly independent if

$$k_1\vec{A}_1 + k_2\vec{A}_2 + \ldots\ldots + k_m\vec{A}_m = \vec{0}$$

has only one solution of

$$k_1 = k_2 = \ldots\ldots = k_m = 0$$

Example 10

Are the three vectors

$$\vec{A}_1 = \begin{bmatrix} 25 \\ 64 \\ 144 \end{bmatrix}, \vec{A}_2 = \begin{bmatrix} 5 \\ 8 \\ 12 \end{bmatrix}, \vec{A}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

linearly independent?

Solution

Writing the linear combination of the three vectors

$$k_1\begin{bmatrix} 25 \\ 64 \\ 144 \end{bmatrix} + k_2\begin{bmatrix} 5 \\ 8 \\ 12 \end{bmatrix} + k_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gives

$$\begin{bmatrix} 25k_1 + 5k_2 + k_3 \\ 64k_1 + 8k_2 + k_3 \\ 144k_1 + 12k_2 + k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The above equations have only one solution, $k_1 = k_2 = k_3 = 0$. However, how do we show that this is the only solution? This is shown below.

The above equations are

$$25k_1 + 5k_2 + k_3 = 0 \qquad (1)$$

$$64k_1 + 8k_2 + k_3 = 0 \qquad (2)$$

$$144k_1 + 12k_2 + k_3 = 0 \qquad (3)$$

Subtracting Eqn (1) from Eqn (2) gives

$$39k_1 + 3k_2 = 0$$

$$k_2 = -13k_1 \qquad (4)$$

Multiplying Eqn (1) by 8 and subtracting it from Eqn (2) that is first multiplied by 5 gives

$$120k_1 - 3k_3 = 0$$

$$k_3 = 40k_1 \qquad (5)$$

Remember we found Eqn (4) and Eqn (5) just from Eqns (1) and (2).

Substitution of Eqns (4) and (5) in Eqn (3) for k_1 and k_2 gives

$$144k_1 + 12(-13k_1) + 40k_1 = 0$$

$$28k_1 = 0$$

$$k_1 = 0$$

This means that k_1 has to be zero, and coupled with (4) and (5), k_2 and k_3 are also zero. So the only solution is $k_1 = k_2 = k_3 = 0$. The three vectors hence are linearly independent.

Example 11

Are the three vectors

$$\vec{A}_1 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \vec{A}_2 = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}, A_3 = \begin{bmatrix} 6 \\ 14 \\ 24 \end{bmatrix}$$

linearly independent?

Solution

By inspection,

$$\vec{A}_3 = 2\vec{A}_1 + 2\vec{A}_2$$

or

$$-2\vec{A}_1 - 2\vec{A}_2 + \vec{A}_3 = \vec{0}$$

So the linear combination

$$k_1\vec{A}_1 + k_2\vec{A}_2 + k_3\vec{A}_3 = \vec{0}$$

has a non-zero solution

$$k_1 = -2,\ k_2 = -2,\ k_3 = 1$$

Hence, the set of vectors is linearly dependent.

What if I cannot prove by inspection, what do I do? Put the linear combination of three vectors equal to the zero vector,

$$k_1\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + k_2\begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} + k_3\begin{bmatrix} 6 \\ 14 \\ 24 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

to give

$$k_1 + 2k_2 + 6k_3 = 0 \qquad (1)$$

$$2k_1 + 5k_2 + 14k_3 = 0 \qquad (2)$$

$$5k_1 + 7k_2 + 24k_3 = 0 \qquad (3)$$

Multiplying Eqn (1) by 2 and subtracting from Eqn (2) gives

$$k_2 + 2k_3 = 0$$

$$k_2 = -2k_3 \qquad (4)$$

Multiplying Eqn (1) by 2.5 and subtracting from Eqn (2) gives

$$-0.5k_1 - k_3 = 0$$

$$k_1 = -2k_3 \qquad (5)$$

Remember we found Eqn (4) and Eqn (5) just from Eqns (1) and (2).

Substitute Eqn (4) and (5) in Eqn (3) for k_1 and k_2 gives

$$5(-2k_3) + 7(-2k_3) + 24k_3 = 0$$

$$-10k_3 - 14k_3 + 24k_3 = 0$$
$$0 = 0$$

This means any values satisfying Eqns (4) and (5) will satisfy Eqns (1), (2) and (3) simultaneously.

For example, chose

$k_3 = 6$, then

$k_2 = -12$ from Eqn (4), and

$k_1 = -12$ from Eqn (5).

Hence we have a nontrivial solution of $[k_1 \quad k_2 \quad k_3] = [-12 \quad -12 \quad 6]$. This implies the three given vectors are linearly dependent. Can you find another nontrivial solution?

What about the following three vectors?

$$\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 6 \\ 14 \\ 25 \end{bmatrix}$$

Are they linearly dependent or linearly independent?

Note that the only difference between this set of vectors and the previous one is the third entry in the third vector. Hence, equations (4) and (5) are still valid. What conclusion do you draw when you plug in equations (4) and (5) in the third equation: $5k_1 + 7k_2 + 25k_3 = 0$? What has changed?

Example 12

Are the three vectors

$$\vec{A}_1 = \begin{bmatrix} 25 \\ 64 \\ 89 \end{bmatrix}, \vec{A}_2 = \begin{bmatrix} 5 \\ 8 \\ 13 \end{bmatrix}, \vec{A}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

linearly independent?

Solution

Writing the linear combination of the three vectors and equating to zero vector

$$k_1 \begin{bmatrix} 25 \\ 64 \\ 89 \end{bmatrix} + k_2 \begin{bmatrix} 5 \\ 8 \\ 13 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gives

$$\begin{bmatrix} 25k_1 + 5k_2 + k_3 \\ 64k_1 + 8k_2 + k_3 \\ 89k_1 + 13k_2 + 2k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In addition to $k_1 = k_2 = k_3 = 0$, one can find other solutions for which k_1, k_2, k_3 are not equal to zero. For example $k_1 = 1, k_2 = -13, k_3 = 40$ is also a solution. This implies

$$1\begin{bmatrix}25\\64\\89\end{bmatrix}-13\begin{bmatrix}5\\8\\13\end{bmatrix}+40\begin{bmatrix}1\\1\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

So the linear combination that gives us a zero vector consists of non-zero constants. Hence $\vec{A}_1,\vec{A}_2,\vec{A}_3$ are linearly dependent.

What do you mean by the rank of a set of vectors?

From a set of n-dimensional vectors, the maximum number of linearly independent vectors in the set is called the rank of the set of vectors. *Note that the rank of the vectors can never be greater than the vectors dimension.*

Example 13

What is the rank of

$$\vec{A}_1=\begin{bmatrix}25\\64\\144\end{bmatrix},\vec{A}_2=\begin{bmatrix}5\\8\\12\end{bmatrix},\vec{A}_3=\begin{bmatrix}1\\1\\1\end{bmatrix}?$$

Solution

Since we found in Example 2.10 that $\vec{A}_1,\vec{A}_2,\vec{A}_3$ are linearly independent, the rank of the set of vectors $\vec{A}_1,\vec{A}_2,\vec{A}_3$ is 3.

Example 14

What is the rank of

$$\vec{A}_1=\begin{bmatrix}25\\64\\89\end{bmatrix},\vec{A}_2=\begin{bmatrix}5\\8\\13\end{bmatrix},\vec{A}_3=\begin{bmatrix}1\\1\\2\end{bmatrix}?$$

Solution

In Example 2.12, we found that $\vec{A}_1,\vec{A}_2,\vec{A}_3$ are linearly dependent, the rank of $\vec{A}_1,\vec{A}_2,\vec{A}_3$ is hence not 3, and is less than 3. Is it 2? Let us choose

$$\vec{A}_1=\begin{bmatrix}25\\64\\89\end{bmatrix},\vec{A}_2=\begin{bmatrix}5\\8\\13\end{bmatrix}$$

Linear combination of $\vec{A}_1$ and $\vec{A}_2$ equal to zero has only one solution. Therefore, the rank is 2.

Example 15

What is the rank of

$$\vec{A}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{A}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \vec{A}_3 = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} ?$$

Solution

From inspection,

$$\vec{A}_2 = 2\vec{A}_1,$$

that implies

$$2\vec{A}_1 - \vec{A}_2 + 0\vec{A}_3 = \vec{0}.$$

Hence

$$k_1\vec{A}_1 + k_2\vec{A}_2 + k_3\vec{A}_3 = \vec{0}.$$

has a nontrivial solution.

So $\vec{A}_1, \vec{A}_2, \vec{A}_3$ are linearly dependent, and hence the rank of the three vectors is not 3. Since

$$\vec{A}_2 = 2\vec{A}_1,$$

$\vec{A}_1$ and $\vec{A}_2$ are linearly dependent, but

$$k_1\vec{A}_1 + k_3\vec{A}_3 = \vec{0}.$$

has trivial solution as the only solution. So $\vec{A}_1$ and $\vec{A}_3$ are linearly independent. The rank of the above three vectors is 2.

Prove that if a set of vectors contains the null vector, the set of vectors is linearly dependent.

Let $\vec{A}_1, \vec{A}_2, ..., \vec{A}_m$ be a set of n-dimensional vectors, then

$$k_1\vec{A}_1 + k_2\vec{A}_2 + \ldots + k_m\vec{A}_m = \vec{0}$$

is a linear combination of the m vectors. Then assuming if $\vec{A}_1$ is the zero or null vector, any value of k_1 coupled with $k_2 = k_3 = \ldots = k_m = 0$ will satisfy the above equation. Hence, the set of vectors is linearly dependent as more than one solution exists.

Prove that if a set of vectors are linearly independent, then a subset of the *m* vectors also has to be linearly independent.

Let this subset be

$$\vec{A}_{a1}, \vec{A}_{a2}, \ldots, \vec{A}_{ap}$$

where $p < m$.

Then if this subset is linearly dependent, the linear combination

$$k_1\vec{A}_{a1} + k_2\vec{A}_{a2} + \ldots + k_p\vec{A}_{ap} = \vec{0}$$

has a non-trivial solution.

So

$$k_1\vec{A}_{a1} + k_2\vec{A}_{a2} + \ldots + k_p\vec{A}_{ap} + 0\vec{A}_{a(p+1)} + ... + 0\vec{A}_{am} = \vec{0}$$

also has a non-trivial solution too, where $\vec{A}_{a(p+1)}, \ldots, \vec{A}_{am}$ are the rest of the $(m-p)$ vectors. However, this is a contradiction. Therefore, a subset of linearly independent vectors cannot be linearly dependent.

Prove that if a set of vectors is linearly dependent, then at least one vector can be written as a linear combination of others.

Let $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_m$ be linearly dependent, then there exists a set of numbers $k_1, \ldots, k_m$ not all of which are zero for the linear combination

$$k_1\vec{A}_1 + k_2\vec{A}_2 + \ldots + k_m\vec{A}_m = \vec{0}.$$

Let $k_p \neq 0$ to give one of the non-zero values of $k_i, i = 1, \ldots, m$, be for $i = p$, then

$$A_p = -\frac{k_2}{k_p}\vec{A}_2 - \ldots - \frac{k_{p-1}}{k_p}\vec{A}_{p-1} - \frac{k_{p+1}}{k_p}\vec{A}_{p+1} - \ldots - \frac{k_m}{k_p}\vec{A}_m.$$

and that proves the theorem.

Prove that if the dimension of a set of vectors is less than the number of vectors in the set, then the set of vectors is linearly dependent.

Can you prove it?

How can vectors be used to write simultaneous linear equations?

If a set of m linear equations with n unknowns is written as

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + \ldots + a_{2n}x_n &= c_2 \\ \vdots \qquad\quad & \vdots \\ \vdots \qquad\quad & \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= c_n \end{aligned}$$

where

$x_1, x_2, \ldots, x_n$ are the unknowns, then in the vector notation they can be written as

$$x_1\vec{A}_1 + x_2\vec{A}_2 + \ldots + x_n\vec{A}_n = \vec{C}$$

where

$$\vec{A}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}$$

where

$$\vec{A}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}$$

$$\vec{A}_2 = \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix}$$

$$\vec{A}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$\vec{C}_1 = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

The problem now becomes whether you can find the scalars $x_1, x_2, \ldots\ldots, x_n$ such that the linear combination

$$x_1\vec{A}_1 + \ldots\ldots\ldots + x_n\vec{A}_n = \vec{C}$$

Example 16

Write

$$25x_1 + 5x_2 + x_3 = 106.8$$
$$64x_1 + 8x_2 + x_3 = 177.2$$
$$144x_1 + 12x_2 + x_3 = 279.2$$

as a linear combination of vectors.

Solution

$$\begin{bmatrix} 25x_1 & +5x_2 & +x_3 \\ 64x_1 & +8x_2 & +x_3 \\ 144x_1 & +12x_2 & +x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$x_1\begin{bmatrix} 25 \\ 64 \\ 144 \end{bmatrix} + x_2\begin{bmatrix} 5 \\ 8 \\ 12 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

What is the definition of the dot product of two vectors?

Let $\vec{A} = [a_1, a_2, \ldots, a_n]$ and $\vec{B} = [b_1, b_2, \ldots, b_n]$ be two n-dimensional vectors. Then the dot product of the two vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + \ldots + a_nb_n = \sum_{i=1}^{n} a_ib_i$$

A dot product is also called an inner product or scalar.

Example 17

Find the dot product of the two vectors $\vec{A}$ = (4, 1, 2, 3) and $\vec{B}$ = (3, 1, 7, 2).

Solution

$$\vec{A} \cdot \vec{B} = (4,1,2,3) \cdot (3,1,7,2)$$
$$= (4)(3)+(1)(1)+(2)(7)+(3)(2)$$
$$= 33$$

Example 18

A product line needs three types of rubber as given in the table below.

Rubber Type	Weight (lbs)	Cost per pound ($)
A	200	20.23
B	250	30.56
C	310	29.12

Use the definition of a dot product to find the total price of the rubber needed.

Solution

The weight vector is given by

$$\vec{W} = (200,250,310)$$

and the cost vector is given by

$$\vec{C} = (20.23,30.56,29.12).$$

The total cost of the rubber would be the dot product of $\vec{W}$ and $\vec{C}$.

$$\begin{aligned}\vec{W} \cdot \vec{C} &= (200,250,310)\cdot(20.23,30.56,29.12)\\ &= (200)(20.23) + (250)(30.56) + (310)(29.12)\\ &= 4046 + 7640 + 9027.2\\ &= \$20713.20\end{aligned}$$

Key Terms:

Vector
Addition of vectors
Rank
Dot Product
Subtraction of vectors
Unit vector
Scalar multiplication of vectors
Null vector
Linear combination of vectors
Linearly independent vectors