# LU Decomposition

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Transforming Numerical Methods Education for STEM Undergraduates

# LU Decomposition

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# LU Decomposition

LU Decomposition is another method to solve a set of simultaneous linear equations

Which is better, Gauss Elimination or LU Decomposition?

To answer this, a closer look at LU decomposition is needed.

# LU Decomposition

## Method

For most non-singular matrix [*A*] that one could conduct Naïve Gauss Elimination forward elimination steps, one can always write it as

$$[A] = [L][U]$$

where

$[L]$ = lower triangular matrix

$[U]$ = upper triangular matrix

# How does LU Decomposition work?

| | |
|---|---|
| If solving a set of linear equations | $[A][X] = [C]$ |
| If $[A] = [L][U]$ then | $[L][U][X] = [C]$ |
| Multiply by | $[L]^{-1}$ |
| Which gives | $[L]^{-1}[L][U][X] = [L]^{-1}[C]$ |
| Remember $[L]^{-1}[L] = [I]$ which leads to | $[I][U][X] = [L]^{-1}[C]$ |
| Now, if $[I][U] = [U]$ then | $[U][X] = [L]^{-1}[C]$ |
| Now, let | $[L]^{-1}[C]=[Z]$ |
| Which ends with | $[L][Z] = [C] \quad (1)$ |
| and | $[U][X] = [Z] \quad (2)$ |

# LU Decomposition

## How can this be used?

Given $[A][X] = [C]$

1. Decompose $[A]$ into $[L]$ and $[U]$
2. Solve $[L][Z] = [C]$ for $[Z]$
3. Solve $[U][X] = [Z]$ for $[X]$

# When is LU Decomposition better than Gaussian Elimination?

To solve $[A][X] = [B]$

**Table.** Time taken by methods

| Gaussian Elimination | LU Decomposition |
|---|---|
| $T\left(\frac{8n^3}{3}+12n^2+\frac{4n}{3}\right)$ | $T\left(\frac{8n^3}{3}+12n^2+\frac{4n}{3}\right)$ |

where T = clock cycle time and n = size of the matrix

So both methods are equally efficient.

# To find inverse of [A]

Time taken by Gaussian Elimination

$$= n\left(CT|_{FE} + CT|_{BS}\right)$$

$$= T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right)$$

Time taken by LU Decomposition

$$= CT|_{LU} + n \times CT|_{FS} + n \times CT|_{BS}$$

$$= T\left(\frac{32n^3}{3} + 12n^2 + \frac{20n}{3}\right)$$

**Table 1** Comparing computational times of finding inverse of a matrix using LU decomposition and Gaussian elimination.

| $n$ | 10 | 100 | 1000 | 10000 |
|---|---|---|---|---|
| $CT\|_{\text{inverse GE}} / CT\|_{\text{inverse LU}}$ | 3.28 | 25.83 | 250.8 | 2501 |

# Method: [A] Decompose to [L] and [U]

$$[A]=[L][U]=\begin{bmatrix}1 & 0 & 0\\ \ell_{21} & 1 & 0\\ \ell_{31} & \ell_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

[*U*] is the same as the coefficient matrix at the end of the forward elimination step.

[*L*] is obtained using the *multipliers* that were used in the forward elimination process

# Finding the [*U*] matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Step 1: $\frac{64}{25} = 2.56; \quad Row2 - Row1(2.56) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$

$$\frac{144}{25} = 5.76; \quad Row3 - Row1(5.76) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

# Finding the [U] Matrix

Matrix after Step 1: $\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$

Step 2: $\dfrac{-16.8}{-4.8} = 3.5; \quad Row3 - Row2(3.5) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$

$$[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

# Finding the [*L*] matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

Using the multipliers used during the Forward Elimination Procedure

From the first step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\ell_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$\ell_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

# Finding the [L] Matrix

From the second step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} \qquad \ell_{32} = \frac{a_{32}}{a_{22}} = \frac{-16.8}{-4.8} = 3.5$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

# Does [L][U] = [A]?

$$[L][U]=\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}= \;?$$

# Example: Unbalanced three phase load

Three-phase loads are common in AC systems. When the system is balanced the analysis can be simplified to a single equivalent circuit model. However, when it is unbalanced the only practical solution involves the solution of simultaneous linear equations. In one model the following equations need to be solved.

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0.4516 & 0.7460 & 0.0080 & 0.0100 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.7787 & -0.5205 & 0.0100 & -0.0080 \\ 0.0080 & 0.0100 & 0.5205 & 0.7787 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.0100 & -0.0080 & 0.8080 & -0.6040 \\ 0.0080 & 0.0100 & 0.0080 & 0.0100 & 0.6040 & 0.8080 \end{bmatrix} \begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 120 \\ 0.000 \\ -60.00 \\ -103.9 \\ -60.00 \\ 103.9 \end{bmatrix}$$

Find the values of $I_{ar}$, $I_{ai}$, $I_{br}$, $I_{bi}$, $I_{cr}$, and $I_{ci}$ using LU Decomposition.

# Example: Unbalanced three phase load

Use Forward Elimination to obtain the [$U$] matrix.

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0.4516 & 0.7460 & 0.0080 & 0.0100 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.7787 & -0.5205 & 0.0100 & -0.0080 \\ 0.0080 & 0.0100 & 0.5205 & 0.7787 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.0100 & -0.0080 & 0.8080 & -0.6040 \\ 0.0080 & 0.0100 & 0.0080 & 0.0100 & 0.6040 & 0.8080 \end{bmatrix}$$

# Example: Unbalanced three phase load

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0.4516 & 0.7460 & 0.0080 & 0.0100 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.7787 & -0.5205 & 0.0100 & -0.0080 \\ 0.0080 & 0.0100 & 0.5205 & 0.7787 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.0100 & -0.0080 & 0.8080 & -0.6040 \\ 0.0080 & 0.0100 & 0.0080 & 0.0100 & 0.6040 & 0.8080 \end{bmatrix}$$

Step 1

for Row 2: $\frac{0.4516}{0.7460} = 0.60536;$

$$Row2 - Row1(0.60536) = [0 \;\; 1.0194 \;\; 0.0019464 \;\; 0.014843 \;\; 0.0019464 \;\; 0.014843]$$

for Row 3: $\frac{0.0100}{0.7460} = 0.013405;$

$$Row3 - Row1(0.13405) = [0 \;\; -0.0019464 \;\; 0.77857 \;\; -0.52061 \;\; 0.0098660 \;\; -0.007893]$$

# Example: Unbalanced three phase load

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0.4516 & 0.7460 & 0.0080 & 0.0100 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.7787 & -0.5205 & 0.0100 & -0.0080 \\ 0.0080 & 0.0100 & 0.5205 & 0.7787 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.0100 & -0.0080 & 0.8080 & -0.6040 \\ 0.0080 & 0.0100 & 0.0080 & 0.0100 & 0.6040 & 0.8080 \end{bmatrix}$$

for Row 4: $\frac{0.0080}{0.7460} = 0.010724;$

$$Row4 - Row1(0.010724) = [0 \;\; 0.014843 \;\; 0.52039 \;\; 0.77879 \;\; 0.0078928 \;\; 0.010086]$$

for Row 5: $\frac{0.0100}{0.7460} = 0.013405;$

$$Row5 - Row1(0.013405) = [0 \;\; -0.0019464 \;\; 0.0098660 \;\; -0.0078928 \;\; 0.80787 \;\; -0.60389]$$

# Example: Unbalanced three phase load

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0.4516 & 0.7460 & 0.0080 & 0.0100 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.7787 & -0.5205 & 0.0100 & -0.0080 \\ 0.0080 & 0.0100 & 0.5205 & 0.7787 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.0100 & -0.0080 & 0.8080 & -0.6040 \\ 0.0080 & 0.0100 & 0.0080 & 0.0100 & 0.6040 & 0.8080 \end{bmatrix}$$

for Row 6: $\dfrac{0.0080}{0.7460} = 0.010724;$

$$Row6 - Row1(0.010724) = [0 \;\; 0.014843 \;\; 0.0078928 \;\; 0.010086 \;\; 0.60389 \;\; 0.80809]$$

# Example: Unbalanced three phase load

The system of equations after the completion of the first step of forward elimination is:

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & -0.0019464 & 0.77857 & -0.52039 & 0.0098660 & -0.0078928 \\ 0 & 0.014843 & 0.52039 & 0.77879 & 0.0078928 & 0.010086 \\ 0 & -0.0019464 & 0.0098660 & -0.0078928 & 0.80787 & -0.60389 \\ 0 & 0.014843 & 0.0078928 & 0.010086 & 0.60389 & 0.80809 \end{bmatrix}$$

# Example: Unbalanced three phase load

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & -0.0019464 & 0.77857 & -0.52039 & 0.0098660 & -0.0078928 \\ 0 & 0.014843 & 0.52039 & 0.77879 & 0.0078928 & 0.010086 \\ 0 & -0.0019464 & 0.0098660 & -0.0078928 & 0.80787 & -0.60389 \\ 0 & 0.014843 & 0.0078928 & 0.010086 & 0.60389 & 0.80809 \end{bmatrix}$$

Step 2

for Row 3: $\dfrac{-0.0019464}{1.0194} = -0.0019094;$

$$Row3 - Row2(-0.0019094) = [0 \;\; 0 \;\; 0.77857 \;\; -0.52036 \;\; 0.0098697 \;\; -0.0078644]$$

for Row 4: $\dfrac{0.014843}{1.0194} = 0.014561;$

$$Row4 - Row2(0.014561) = [0 \;\; 0 \;\; 0.52036 \;\; 0.77857 \;\; 0.0078644 \;\; 0.0098697]$$

# Example: Unbalanced three phase load

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & -0.0019464 & 0.77857 & -0.52039 & 0.0098660 & -0.0078928 \\ 0 & 0.014843 & 0.52039 & 0.77879 & 0.0078928 & 0.010086 \\ 0 & -0.0019464 & 0.0098660 & -0.0078928 & 0.80787 & -0.60389 \\ 0 & 0.014843 & 0.0078928 & 0.010086 & 0.60389 & 0.80809 \end{bmatrix}$$

for Row 5: $\frac{-0.0019464}{1.0194} =; -0.0019094;$

$$Row5 - Row2(-0.0019094) = [0 \;\; 0 \;\; 0.0098697 \;\; -0.0078644 \;\; 0.80787 \;\; -0.60386]$$

for Row 6: $\frac{0.014843}{1.0194} = 0.014561;$

$$Row6 - Row2(0.014561) = [0 \;\; 0 \;\; 0.0078644 \;\; 0.0098697 \;\; 0.60386 \;\; 0.80787]$$

# Example: Unbalanced three phase load

The system of equations after the completion of the second step of forward elimination is:

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & 0 & 0.77857 & -0.52036 & 0.0098697 & -0.0078644 \\ 0 & 0 & 0.52036 & 0.77857 & 0.0078644 & 0.0098697 \\ 0 & 0 & 0.0098697 & -0.0078644 & 0.80787 & -0.60386 \\ 0 & 0 & 0.0078644 & 0.0098697 & 0.60386 & 0.80787 \end{bmatrix}$$

# Example: Unbalanced three phase load

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & 0 & 0.77857 & -0.52036 & 0.0098697 & -0.0078644 \\ 0 & 0 & 0.52036 & 0.77857 & 0.0078644 & 0.0098697 \\ 0 & 0 & 0.0098697 & -0.0078644 & 0.80787 & -0.60386 \\ 0 & 0 & 0.0078644 & 0.0098697 & 0.60386 & 0.80787 \end{bmatrix}$$

Step 3

for Row 4: $\frac{0.52036}{0.77857} = 0.66836;$

$$Row4 - Row3(0.66836) = [0 \;\; 0 \;\; 0 \;\; 1.1264 \;\; 0.0012679 \;\; 0.015126]$$

for Row 5: $\frac{0.0098697}{0.77857} = 0.012677;$

$$Row5 - Row3(0.012677) = [0 \;\; 0 \;\; 0 \;\; -0.0012679 \;\; 0.807745 \;\; -0.60376]$$

# Example: Unbalanced three phase load

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & 0 & 0.77857 & -0.52036 & 0.0098697 & -0.0078644 \\ 0 & 0 & 0.52036 & 0.77857 & 0.0078644 & 0.0098697 \\ 0 & 0 & 0.0098697 & -0.0078644 & 0.80787 & -0.60386 \\ 0 & 0 & 0.0078644 & 0.0098697 & 0.60386 & 0.80787 \end{bmatrix}$$

for Row 6: $\dfrac{0.0078644}{0.77857} = 0.01010;$

$$Row6 - Row3(0.01010) = \begin{bmatrix} 0 & 0 & 0 & 0.015126 & 0.60376 & 0.80795 \end{bmatrix}$$

# Example: Unbalanced three phase load

The system of equations after the completion of the third step of forward elimination is:

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & 0 & 0.77857 & -0.52036 & 0.0098697 & -0.0078644 \\ 0 & 0 & 0 & 1.1264 & 0.0012679 & 0.015126 \\ 0 & 0 & 0 & -0.0012679 & 0.807745 & -0.60376 \\ 0 & 0 & 0 & 0.015126 & 0.60376 & 0.80795 \end{bmatrix}$$

# Example: Unbalanced three phase load

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & 0 & 0.77857 & -0.52036 & 0.0098697 & -0.0078644 \\ 0 & 0 & 0 & 1.1264 & 0.0012679 & 0.015126 \\ 0 & 0 & 0 & -0.0012679 & 0.807745 & -0.60376 \\ 0 & 0 & 0 & 0.015126 & 0.60376 & 0.80795 \end{bmatrix}$$

Step 4

for Row 5: $\frac{-0.0012679}{1.1264} = -0.0011257;$

$$Row5 - Row4(-0.0011257) = [0\ \ 0\ \ 0\ \ 0\ \ 0.80775\ \ -0.60375]$$

for Row 6: $\frac{0.015126}{1.1264} = 0.013429;$

$$Row6 - Row4(0.013429) = [0\ \ 0\ \ 0\ \ 0\ \ 0.60375\ \ 0.80775]$$

# Example: Unbalanced three phase load

The system of equations after the completion of the fourth step of forward elimination is:

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & 0 & 0.77857 & -0.52036 & 0.0098697 & -0.0078644 \\ 0 & 0 & 0 & 1.1264 & 0.0012679 & 0.015126 \\ 0 & 0 & 0 & 0 & 0.80775 & -0.60375 \\ 0 & 0 & 0 & 0 & 0.60375 & 0.80775 \end{bmatrix}$$

# Example: Unbalanced three phase load

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & 0 & 0.77857 & -0.52036 & 0.0098697 & -0.0078644 \\ 0 & 0 & 0 & 1.1264 & 0.0012679 & 0.015126 \\ 0 & 0 & 0 & 0 & 0.80775 & -0.60375 \\ 0 & 0 & 0 & 0 & 0.60375 & 0.80775 \end{bmatrix}$$

Step 5

for Row 6: $\frac{0.60375}{0.80775} = 0.74745;$

$$Row6 - Row5(0.74745) = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1.2590]$$

# Example: Unbalanced three phase load

The coefficient matrix at the end of the forward elimination process is the [*U*] matrix

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & 0 & 0.77857 & -0.52036 & 0.0098697 & -0.0078644 \\ 0 & 0 & 0 & 1.1264 & 0.0012679 & 0.015126 \\ 0 & 0 & 0 & 0 & 0.80775 & -0.60375 \\ 0 & 0 & 0 & 0 & 0 & 1.2590 \end{bmatrix}$$

# Example: Unbalanced three phase load

For a system of six equations, the [*L*] matrix is in the form

$$[L] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 & 0 & 0 \\ \ell_{31} & \ell_{32} & 1 & 0 & 0 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 & 0 & 0 \\ \ell_{51} & \ell_{52} & \ell_{53} & \ell_{54} & 1 & 0 \\ \ell_{61} & \ell_{62} & \ell_{63} & \ell_{64} & \ell_{65} & 1 \end{bmatrix}$$

Values of the [*L*] matrix are the multipliers used during the Forward Elimination Procedure

# Example: Unbalanced three phase load

From the first step of forward elimination

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0.4516 & 0.7460 & 0.0080 & 0.0100 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.7787 & -0.5205 & 0.0100 & -0.0080 \\ 0.0080 & 0.0100 & 0.5205 & 0.7787 & 0.0080 & 0.0100 \\ 0.0100 & -0.0080 & 0.0100 & -0.0080 & 0.8080 & -0.6040 \\ 0.0080 & 0.0100 & 0.0080 & 0.0100 & 0.6040 & 0.8080 \end{bmatrix}$$

$$\ell_{21} = \frac{0.4516}{0.7460} = 0.60536$$

$$\ell_{31} = \frac{0.0100}{0.7460} = 0.013405$$

$$\ell_{41} = \frac{0.0080}{0.7460} = 0.010724$$

$$\ell_{51} = \frac{0.0100}{0.7460} = 0.013405$$

$$\ell_{61} = \frac{0.0080}{0.7460} = 0.010724$$

# Example: Unbalanced three phase load

From the second step of forward elimination

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & -0.0019464 & 0.77857 & -0.52039 & 0.0098660 & -0.0078928 \\ 0 & 0.014843 & 0.52039 & 0.77879 & 0.0078928 & 0.010086 \\ 0 & -0.0019464 & 0.0098660 & -0.0078928 & 0.80787 & -0.60389 \\ 0 & 0.014843 & 0.0078928 & 0.010086 & 0.60389 & 0.80809 \end{bmatrix}$$

$$\ell_{32} = \frac{-0.0019464}{1.0194} = -0.0019094$$

$$\ell_{42} = \frac{0.014843}{1.0194} = 0.014561$$

$$\ell_{52} = \frac{-0.0019464}{1.0194} = -0.0019094$$

$$\ell_{62} = \frac{0.014843}{1.0194} = 0.014561$$

# Example: Unbalanced three phase load

From the third step of forward elimination

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & 0 & 0.77857 & -0.52036 & 0.0098697 & -0.0078644 \\ 0 & 0 & 0.52036 & 0.77857 & 0.0078644 & 0.0098697 \\ 0 & 0 & 0.0098697 & -0.0078644 & 0.80787 & -0.60386 \\ 0 & 0 & 0.0078644 & 0.0098697 & 0.60386 & 0.80787 \end{bmatrix}$$

$$\ell_{43} = \frac{0.52036}{0.77857} = 0.66836$$

$$\ell_{53} = \frac{0.0098697}{0.77857} = 0.012677$$

$$\ell_{63} = \frac{0.0078644}{0.77857} = 0.01010$$

# Example: Unbalanced three phase load

From the fourth step of forward elimination

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & 0 & 0.77857 & -0.52036 & 0.0098697 & -0.0078644 \\ 0 & 0 & 0 & 1.1264 & 0.0012679 & 0.015126 \\ 0 & 0 & 0 & -0.0012679 & 0.80774 & -0.60376 \\ 0 & 0 & 0 & 0.015126 & 0.60376 & 0.80795 \end{bmatrix}$$

$$\ell_{54} = \frac{-0.0012679}{1.1264} = -0.0011257$$

$$\ell_{64} = \frac{0.015126}{1.1264} = 0.013429$$

# Example: Unbalanced three phase load

From the fifth step of forward elimination

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & 0 & 0.77857 & -0.52036 & 0.0098697 & -0.0078644 \\ 0 & 0 & 0 & 1.1264 & 0.0012679 & 0.015126 \\ 0 & 0 & 0 & 0 & 0.80775 & -0.60375 \\ 0 & 0 & 0 & 0 & 0.60375 & 0.80775 \end{bmatrix}$$

$$\ell_{65} = \frac{0.60375}{0.80775} = 0.74745$$

# Example: Unbalanced three phase load

The $[L]$ matrix is

$$[L]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.60536 & 1 & 0 & 0 & 0 & 0 \\ 0.013405 & -0.0019094 & 1 & 0 & 0 & 0 \\ 0.010724 & 0.014561 & 0.66836 & 1 & 0 & 0 \\ 0.013405 & -0.0019094 & 0.012677 & -0.0011257 & 1 & 0 \\ 0.010724 & 0.014561 & 0.01010 & 0.013429 & 0.74745 & 1 \end{bmatrix}$$

# Example: Unbalanced three phase load

Does [L][U] = [A]?

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.60536 & 1 & 0 & 0 & 0 & 0 \\ 0.013405 & -0.0019094 & 1 & 0 & 0 & 0 \\ 0.010724 & 0.014561 & 0.66836 & 1 & 0 & 0 \\ 0.013405 & -0.0019094 & 0.012677 & -0.0011257 & 1 & 0 \\ 0.010724 & 0.014561 & 0.01010 & 0.013429 & 0.74745 & 1 \end{bmatrix} \begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & 0 & 0.77857 & -0.52036 & 0.0098697 & -0.0078644 \\ 0 & 0 & 0 & 1.1264 & 0.0012679 & 0.015126 \\ 0 & 0 & 0 & 0 & 0.80775 & -0.60375 \\ 0 & 0 & 0 & 0 & 0 & 1.2590 \end{bmatrix} = ?$$

# Example: Unbalanced three phase load

Set [L][Z] = [C]

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.60536 & 1 & 0 & 0 & 0 & 0 \\ 0.013405 & -0.0019094 & 1 & 0 & 0 & 0 \\ 0.010724 & 0.014561 & 0.66836 & 1 & 0 & 0 \\ 0.013405 & -0.0019094 & 0.012677 & -0.0011257 & 1 & 0 \\ 0.010724 & 0.014561 & 0.01010 & 0.013429 & 0.74745 & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \end{bmatrix} = \begin{bmatrix} 120 \\ 0.000 \\ -60.00 \\ -103.9 \\ -60.00 \\ 103.9 \end{bmatrix}$$

# Example: Unbalanced three phase load

Solve for [Z]

The six equations become

$$z_1 = 120$$

$$0.60536 z_1 + z_2 = 0.00$$

$$0.013405 z_1 + (-0.0019094) z_2 + z_3 = -60.00$$

$$0.010724 z_1 + 0.014561 z_2 + 0.66836 z_3 + z_4 = -103.9$$

$$0.013405 z_1 + (-0.0019094) z_2 + 0.012677 z_3 + (-0.0011257) z_4 + z_5 = -60.00$$

$$0.010724 z_1 + 0.014561 z_2 + 0.01010 z_3 + 0.013429 z_4 + 074745 z_5 + z_6 = 103.9$$

# Example: Unbalanced three phase load

Solve for [Z]

$$
\begin{aligned}
z_1 &= 120 \\
z_2 &= 0.00 - 0.60536 z_1 \\
&= -72.643 \\
z_3 &= -60.00 - 0.013405 z_1 - (-0.0019094) z_2 \\
&= -61.747 \\
z_4 &= -103.9 - 0.010724 z_1 - 0.014561 z_2 - 0.66836 z_3 \\
&= -62.860 \\
z_5 &= -60.00 - 0.013405 z_1 - (-0.0019094) z_2 - 0.012677 z_3 - (-0.0011257) z_4 \\
&= -61.035 \\
z_6 &= 103.9 - 0.010724 z_1 - 0.014561 z_2 - 0.01010 z_3 - 0.013429 z_4 - 074745 z_5 \\
&= 150.76
\end{aligned}
$$

# Example: Unbalanced three phase load

The [$Z$] matrix is

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} = \begin{bmatrix} 120 \\ -72.643 \\ -61.747 \\ -62.860 \\ -61.035 \\ 150.76 \end{bmatrix}$$

# Example: Unbalanced three phase load

Set [*U*] [*I*] = [*Z*]

$$\begin{bmatrix} 0.7460 & -0.4516 & 0.0100 & -0.0080 & 0.0100 & -0.0080 \\ 0 & 1.0194 & 0.0019464 & 0.014843 & 0.0019464 & 0.014843 \\ 0 & 0 & 0.77857 & -0.52036 & 0.0098697 & -0.0078644 \\ 0 & 0 & 0 & 1.1264 & 0.0012679 & 0.015126 \\ 0 & 0 & 0 & 0 & 0.80775 & -0.60375 \\ 0 & 0 & 0 & 0 & 0 & 1.2590 \end{bmatrix} \begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 120 \\ -72.643 \\ -61.747 \\ -62.860 \\ -61.035 \\ 150.76 \end{bmatrix}$$

# Example: Unbalanced three phase load

Solve for [I]

The six equations become

$$0.7460 I_{ar} + (-0.4516) I_{ai} + 0.0100 I_{br} + (-0.0080) I_{bi} + 0.0100 I_{cr} + (-0.0080) I_{ci} = 120$$

$$1.0194 I_{ai} + 0.0019464 I_{br} + 0.014843 I_{bi} + 0.0019464 I_{cr} + 0.014843 I_{ci} = -72.643$$

$$0.77857 I_{br} + (-0.52036) I_{bi} + 0.0098697 I_{cr} + (-0.0078644) I_{ci} = -61.747$$

$$1.1264 I_{bi} + 0.0012679 I_{cr} + 0.015126 I_{ci} = -62.860$$

$$0.80775 I_{cr} + (-0.603748) I_{ci} = -61.035$$

$$1.2590 I_{ci} = 150.76$$

# Example: Unbalanced three phase load

Solve for [I]

Remember to start with the last equation

$$I_{ci} = \frac{150.76}{1.2590} = 119.74$$

$$I_{cr} = \frac{-61.035 - (-0.60375)I_{ci}}{0.80775} = 13.940$$

$$I_{bi} = \frac{-62.860 - 0.0012679 I_{cr} - 0.015126 I_{ci}}{1.1264} = -57.432$$

$$I_{br} = \frac{-61.747 - (-0.52036)I_{bi} - 0.0098697 I_{cr} - (-0.0078644)I_{ci}}{0.77857} = -116.66$$

# Example: Unbalanced three phase load

$$I_{ai} = \frac{-72.643 - 0.0019464 I_{br} - 0.014843 I_{bi} - 0.0019464 I_{cr} - 0.014843 I_{ci}}{1.0194} = -71.973$$

$$I_{ar} = \frac{120 - (-0.4516) I_{ai} - 0.0100 I_{br} - (-0.0080) I_{bi} - 0.0100 I_{cr} - (-0.0080) I_{ci}}{0.7460} = 119.33$$

Solution:

$$\begin{bmatrix} I_{ar} \\ I_{ai} \\ I_{br} \\ I_{bi} \\ I_{cr} \\ I_{ci} \end{bmatrix} = \begin{bmatrix} 119.3331 \\ -71.97344 \\ -116.6607 \\ -57.43159 \\ 13.93977 \\ 119.7439 \end{bmatrix}$$

# Finding the inverse of a square matrix

The inverse [B] of a square matrix [*A*] is defined as

$$[A][B] = [I] = [B][A]$$

# Finding the inverse of a square matrix

How can LU Decomposition be used to find the inverse?

Assume the first column of [*B*] to be $[b_{11}\ \ b_{12}\ \ \ldots\ b_{n1}]^T$

Using this and the definition of matrix multiplication

First column of [*B*]

$$[A]\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Second column of [*B*]

$$[A]\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

The remaining columns in [*B*] can be found in the same manner

# Example: Inverse of a Matrix

Find the inverse of a square matrix [*A*]

$$[A]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}$$

Using the decomposition procedure, the [*L*] and [*U*] matrices are found to be

$$[A]=[L][U]=\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}$$

# Example: Inverse of a Matrix

Solving for the each column of [*B*] requires two steps

1) Solve [*L*] [*Z*] = [*C*] for [*Z*]

2) Solve [*U*] [*X*] = [*Z*] for [*X*]

Step 1: $$[L][Z]=[C]\rightarrow\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

This generates the equations:

$$z_1=1$$

$$2.56z_1+z_2=0$$

$$5.76z_1+3.5z_2+z_3=0$$

# Example: Inverse of a Matrix

Solving for [*Z*]

$$z_1 = 1$$

$$\begin{aligned} z_2 &= 0 - 2.56 z_1 \\ &= 0 - 2.56(1) \\ &= -2.56 \end{aligned}$$

$$\begin{aligned} z_3 &= 0 - 5.76 z_1 - 3.5 z_2 \\ &= 0 - 5.76(1) - 3.5(-2.56) \\ &= 3.2 \end{aligned}$$

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

# Example: Inverse of a Matrix

Solving $[U][X] = [Z]$ for $[X]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$25b_{11} + 5b_{21} + b_{31} = 1$$

$$-4.8b_{21} - 1.56b_{31} = -2.56$$

$$0.7b_{31} = 3.2$$

# Example: Inverse of a Matrix

Using Backward Substitution

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$b_{21} = \frac{-2.56 + 1.560 b_{31}}{-4.8}$$

$$= \frac{-2.56 + 1.560(4.571)}{-4.8} = -0.9524$$

$$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25}$$

$$= \frac{1 - 5(-0.9524) - 4.571}{25} = 0.04762$$

So the first column of the inverse of [*A*] is:

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

# Example: Inverse of a Matrix

Repeating for the second and third columns of the inverse

Second Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

Third Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

# Example: Inverse of a Matrix

The inverse of [*A*] is

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

To check your work do the following operation

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$

# Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/lu_decomposition.html

# THE END

http://numericalmethods.eng.usf.edu