Lagrangian Interpolation

Electrical Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

Lagrange Method of Interpolation

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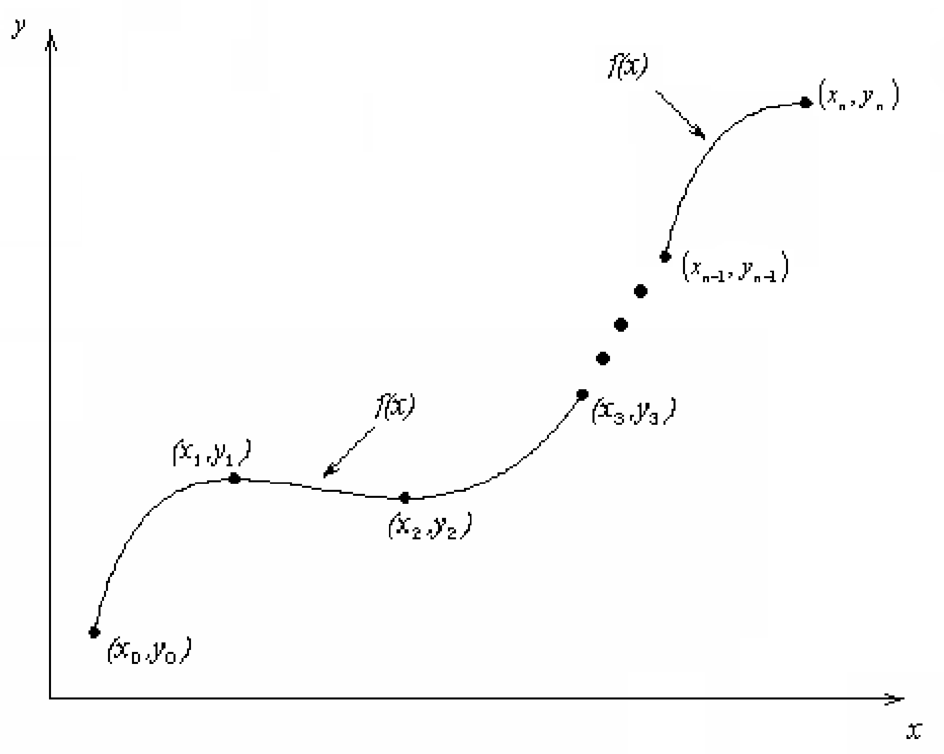
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

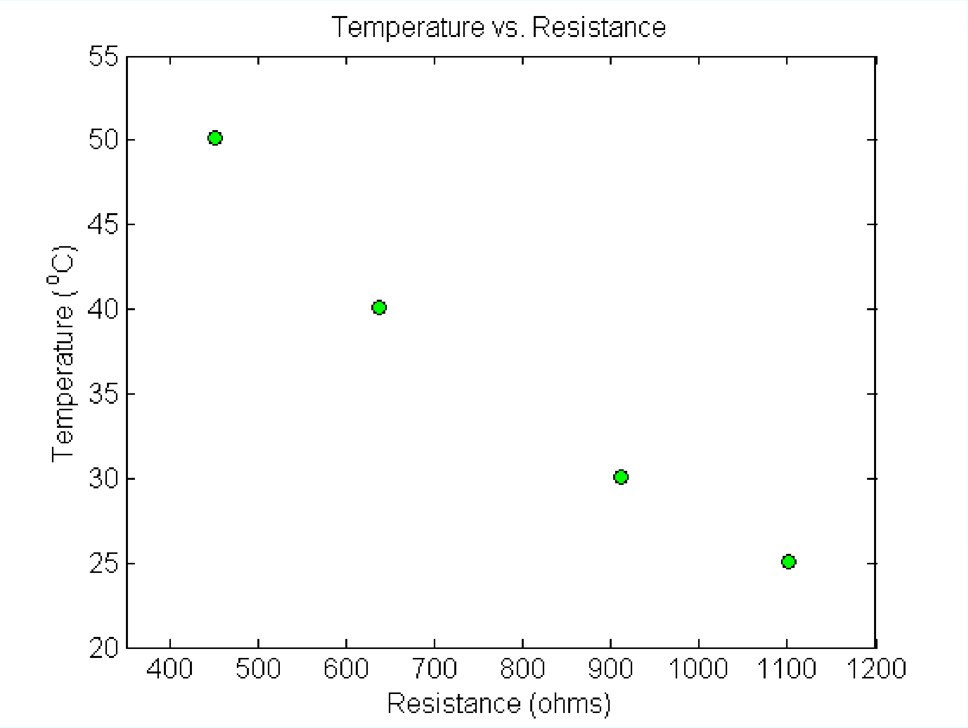
$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Example

Thermistors are based on materials' change in resistance with temperature. A manufacturer of thermistors makes the following observations on a thermistor. Determine the temperature corresponding to 754.8 ohms using the Lagrangian method for linear interpolation.

R (Ω)	T(C)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128



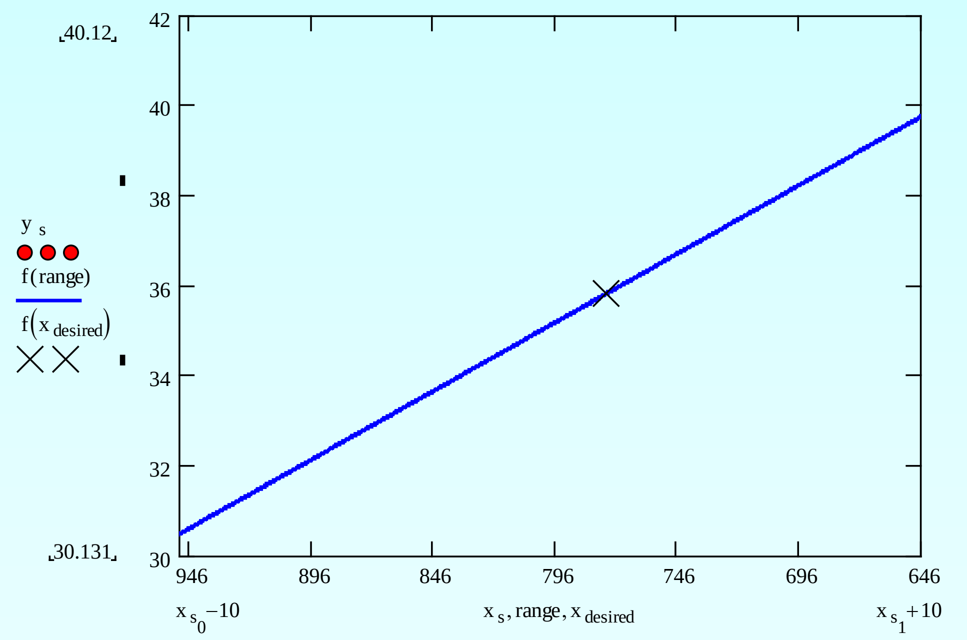
Linear Interpolation

$$T(R) = \sum_{i=0}^{1} L_i(R)T(R_i)$$

$$= L_0(R)T(R_0) + L_1(R)T(R_1)$$

$$R_0 = 911.3, T(R_0) = 30.131$$

$$R_1 = 636.0, T(R_1) = 40.120$$



Linear Interpolation (contd)

$$L_0(R) = \prod_{\substack{j=0 \\ j\neq 0}}^{1} \frac{R - R_j}{R_0 - R_j} = \frac{R - R_1}{R_0 - R_1}$$

$$L_1(R) = \prod_{\substack{j=0 \\ j\neq 1}}^{1} \frac{R - R_j}{R_1 - R_j} = \frac{R - R_0}{R_1 - R_0}$$

$$T(R) = \frac{R - R_1}{R_0 - R_1} T(R_0) + \frac{R - R_0}{R_1 - R_0} T(R_1)$$

$$= \frac{R - 636.0}{911.3 - 636.0}(30.131) + \frac{R - 911.3}{636.0 - 911.3}(40.120)\text{ , } 636.0 \le R \le 911.3$$

$$T(754.8) = \frac{754.8 - 636.0}{911.3 - 636.0}(30.131) + \frac{754.8 - 911.3}{636.0 - 911.3}(40.120)$$

$$= 0.43153(30.131) + 0.56847(40.120)$$

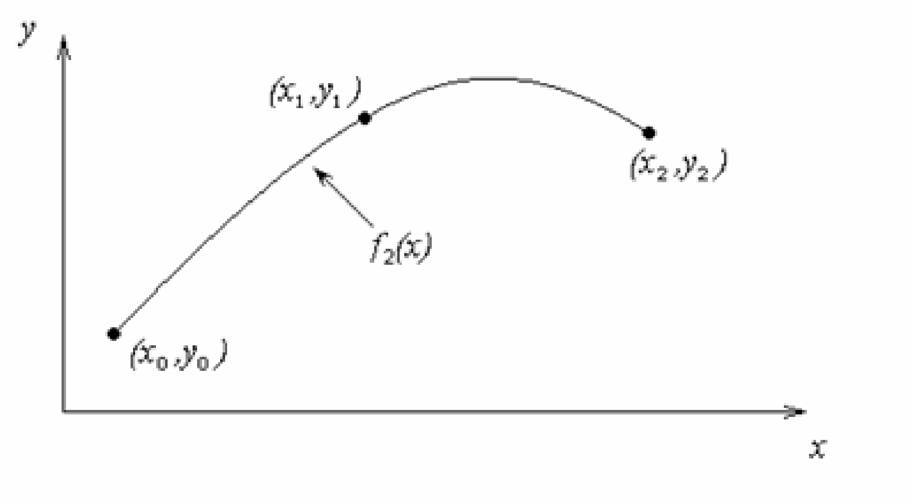
$$= 35.809^\circ C$$

Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the velocity given by

$$v(t) = \sum_{i=0}^{2} L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2)$$



Example

Thermistors are based on materials' change in resistance with temperature. A manufacturer of thermistors makes the following observations on a thermistor. Determine the temperature corresponding to 754.8 ohms using the Lagrangian method for quadratic interpolation.

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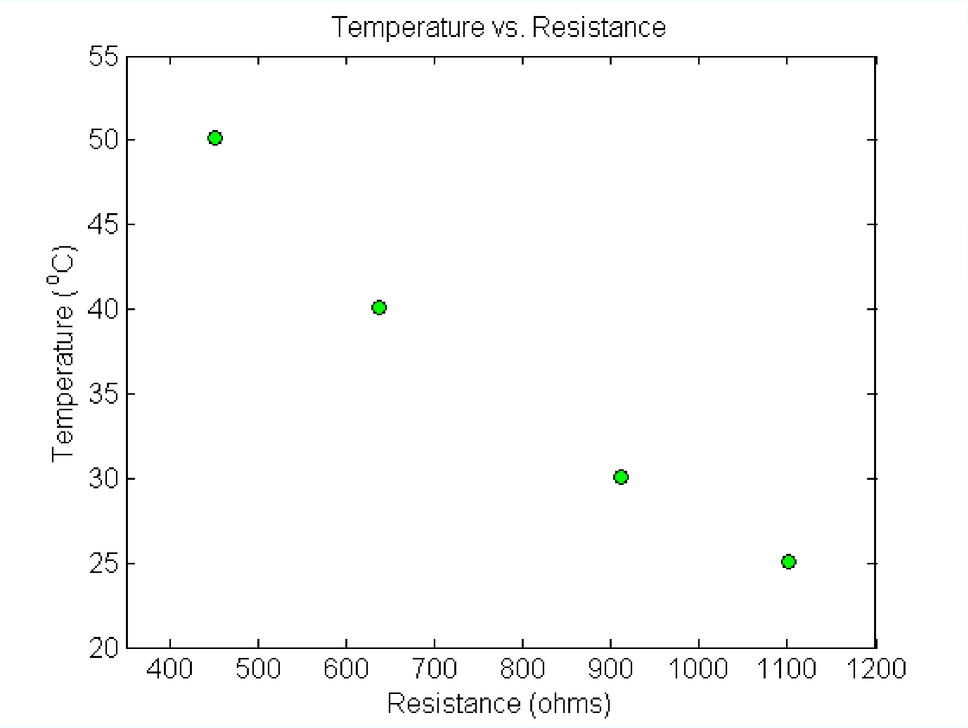
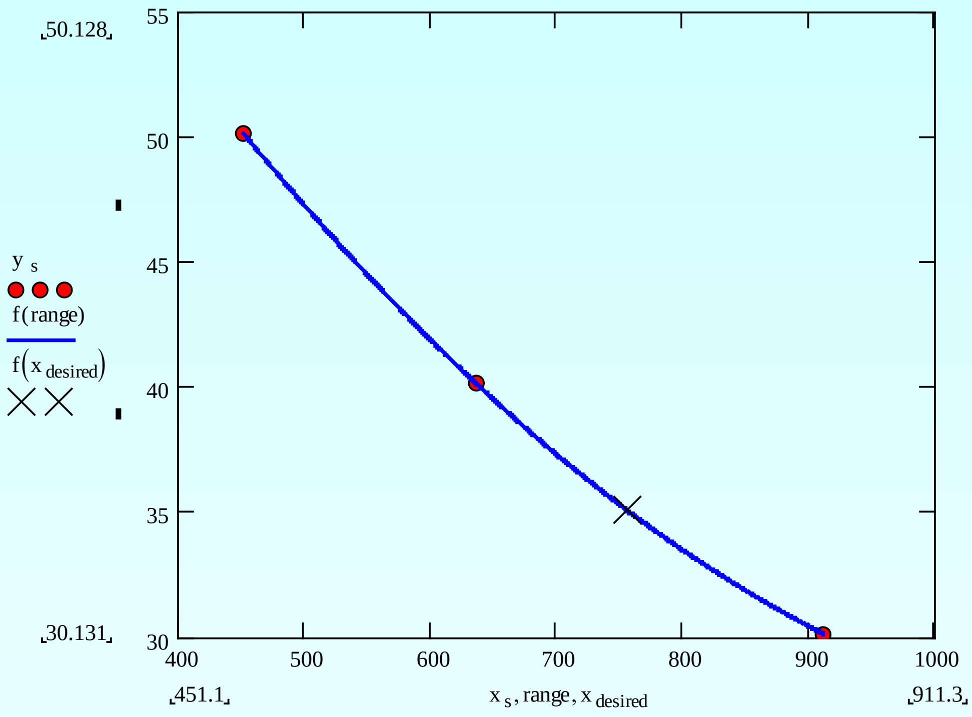


Quadratic Interpolation (contd)

$$R_o = 911.3, \ T(R_o) = 30.131$$

$$R_1 = 636.0, \ T(R_1) = 40.120$$

$$R_2 = 451.1, \ T(R_2) = 50.128$$



$$L_0(R) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{R - R_j}{R_0 - R_j} = \left(\frac{R - R_1}{R_0 - R_1}\right)\left(\frac{R - R_2}{R_0 - R_2}\right)$$

$$L_1(R) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{R - R_j}{R_1 - R_j} = \left(\frac{R - R_0}{R_1 - R_0}\right)\left(\frac{R - R_2}{R_1 - R_2}\right)$$

$$L_2(R) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{R - R_j}{R_2 - R_j} = \left(\frac{R - R_0}{R_2 - R_0}\right)\left(\frac{R - R_1}{R_2 - R_1}\right)$$

Quadratic Interpolation (contd)

$$T(R)=\left(\frac{R-R_1}{R_0-R_1}\right)\left(\frac{R-R_2}{R_0-R_2}\right)T(R_0)+\left(\frac{R-R_0}{R_1-R_0}\right)\left(\frac{R-R_2}{R_1-R_2}\right)T(R_1)+\left(\frac{R-R_0}{R_2-R_0}\right)\left(\frac{R-R_1}{R_2-R_1}\right)T(R_2)$$

$$T(754.8)=\frac{(754.8-636.0)(754.8-451.1)}{(911.3-636.0)(911.3-451.1)}(30.131)+\frac{(754.8-911.3)(754.8-451.1)}{(636.0-911.3)(636.0-451.1)}(40.120)$$
$$+\frac{(754.8-911.3)(754.8-636.0)}{(451.1-911.3)(451.1-636.0)}(50.128)$$

$$=(0.28478)(30.131)+(0.93372)(40.120)+(-0.21850)(50.128)$$

$$=35.089°C$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a|=\left|\frac{35.089-35.809}{35.089}\right|\times 100$$

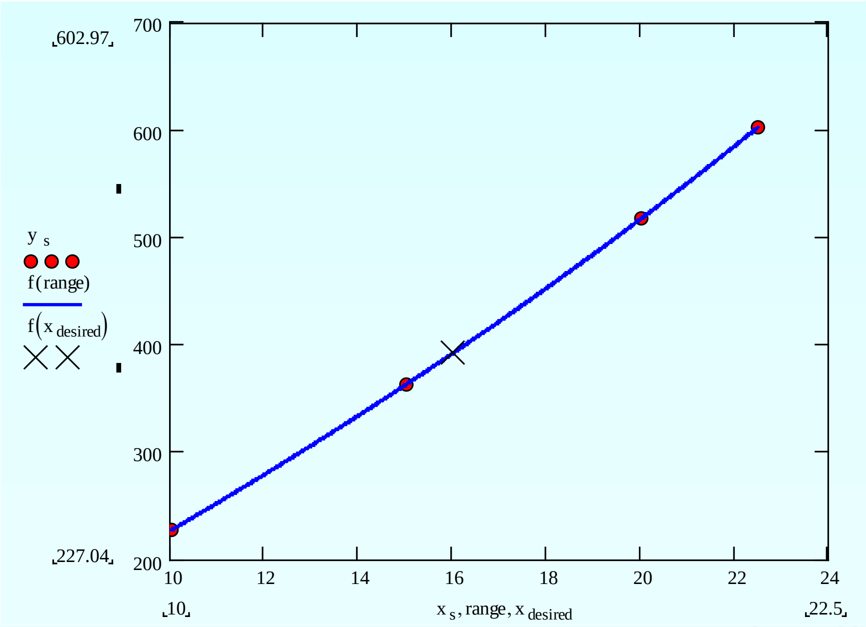
$$=2.0543\%$$

Cubic Interpolation

For the third order polynomial (also called cubic interpolation), we choose the velocity given by

$$v(t) = \sum_{i=0}^{3} L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2) + L_3(t)v(t_3)$$



Example

Thermistors are based on materials' change in resistance with temperature. A manufacturer of thermistors makes the following observations on a thermistor. Determine the temperature corresponding to 754.8 ohms using the Lagrangian method for cubic interpolation.

R (Ω)	T(C)
1101.0	25.113
911.3	30.131
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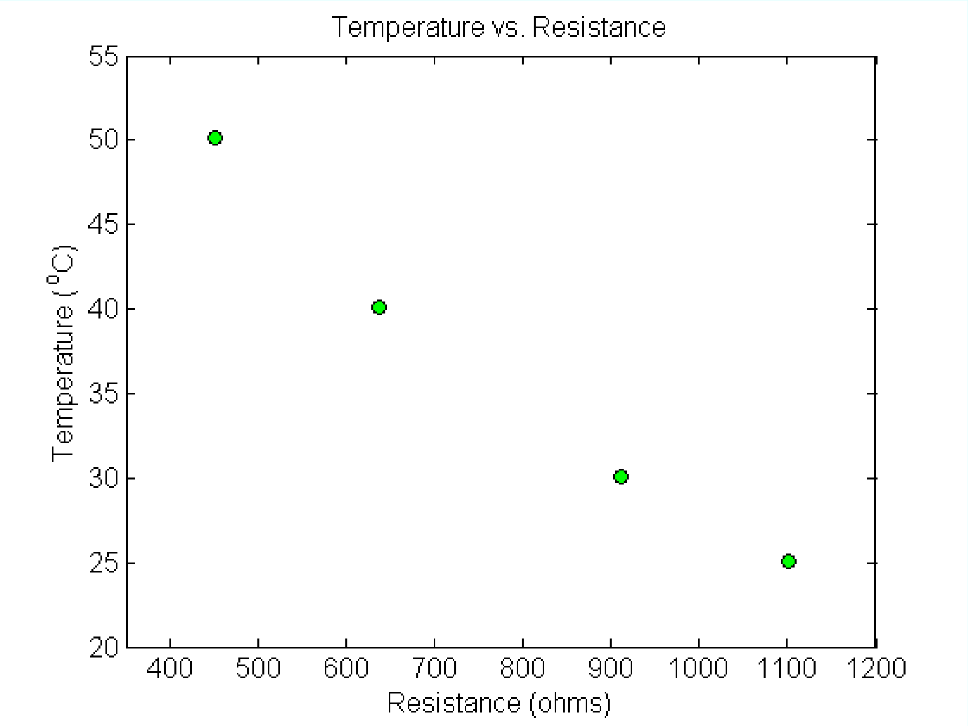


Cubic Interpolation (contd)

$R_o = 1101.0, \; T(R_o) = 25.113$ $R_1 = 911.3, \; T(R_1) = 30.131$

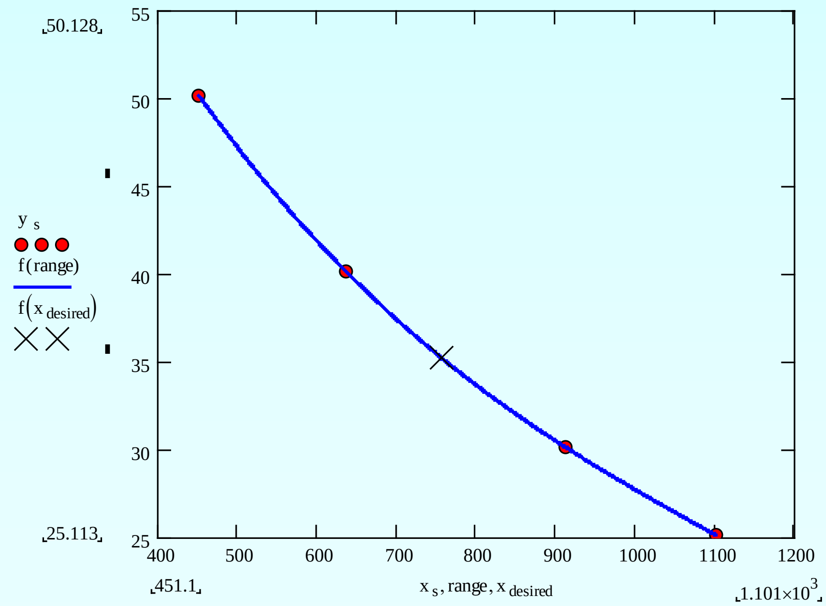
$R_2 = 636.0, \; T(R_2) = 40.120$ $R_3 = 451.1, \; T(R_3) = 50.128$

$$L_0(R) = \prod_{\substack{j=0 \\ j\neq 0}}^{3} \frac{R - R_j}{R_0 - R_j} = \left(\frac{R - R_1}{R_0 - R_1}\right)\left(\frac{R - R_2}{R_0 - R_2}\right)\left(\frac{R - R_3}{R_0 - R_3}\right)$$

$$L_1(R) = \prod_{\substack{j=0 \\ j\neq 1}}^{3} \frac{R - R_j}{R_1 - R_j} = \left(\frac{R - R_0}{R_1 - R_0}\right)\left(\frac{R - R_2}{R_1 - R_2}\right)\left(\frac{R - R_3}{R_1 - R_3}\right)$$

$$L_2(R) = \prod_{\substack{j=0 \\ j\neq 2}}^{3} \frac{R - R_j}{R_2 - R_j} = \left(\frac{R - R_0}{R_2 - R_0}\right)\left(\frac{R - R_1}{R_2 - R_1}\right)\left(\frac{R - R_3}{R_2 - R_3}\right)$$

$$L_3(R) = \prod_{\substack{j=0 \\ j\neq 3}}^{3} \frac{R - R_j}{R_3 - R_j} = \left(\frac{R - R_0}{R_3 - R_0}\right)\left(\frac{R - R_1}{R_3 - R_1}\right)\left(\frac{R - R_2}{R_3 - R_2}\right)$$



Cubic Interpolation (contd)

$$T(R)=\left(\frac{R-R_1}{R_0-R_1}\right)\left(\frac{R-R_2}{R_0-R_2}\right)\left(\frac{R-R_3}{R_0-R_3}\right)T(R_0)+\left(\frac{R-R_0}{R_1-R_0}\right)\left(\frac{R-R_2}{R_1-R_2}\right)\left(\frac{R-R_3}{R_1-R_3}\right)T(R_1)$$

$$+\left(\frac{R-R_0}{R_2-R_0}\right)\left(\frac{R-R_1}{R_2-R_1}\right)\left(\frac{R-R_3}{R_2-R_3}\right)T(R_2)+\left(\frac{R-R_0}{R_3-R_0}\right)\left(\frac{R-R_1}{R_3-R_1}\right)\left(\frac{R-R_2}{R_3-R_2}\right)T(R_3)$$

$$R_0 \le R \le R_0$$

$$T(754.8)=\frac{(754.8-911.3)(754.8-636.0)(754.8-451.1)}{(1101.0-911.3)(1101.0-636.0)(1101.0-451.1)}(25.113)$$

$$+\frac{(754.8-1101.0)(754.8-636.0)(754.8-451.1)}{(911.3-1101.0)(911.3-636.0)(911.3-451.1)}(30.131)$$

$$+\frac{(754.8-1101.0)(754.8-911.3)(754.8-451.1)}{(636.0-1101.0)(636.0-911.3)(636.0-451.1)}(40.120)$$

$$+\frac{(754.8-1101.0)(754.8-911.3)(754.8-636.0)}{(451.1-1101.0)(451.1-911.3)(451.1-636.0)}(50.128)$$

$$=(-0.098494)(25.113)+(0.51972)(30.131)+(0.69517)(40.120)+(-0.11639)(50.128)$$

$$=35.242^{\circ}\mathrm{C}$$

Cubic Interpolation

The absolute percentage relative approximate error, $\left|\in_a\right|$ between second and third order polynomial is

$$\left|\in_a\right| = \left|\frac{35.242 - 35.089}{35.242}\right| \times 100$$

$$= 0.43458\%$$

Comparison Table

Order of Polynomial	1	2	3
Temperature $°C$	35.809	35.089	35.242
Absolute Relative Approximate Error	----------	2.0543%	0.43458%

Actual Calibration

The actual calibration curve used by industry is given by

$$\frac{1}{T}=\sum_{i=0}^{3}L_i(\ln R)\frac{1}{T(\ln R_i)} = L_0(\ln R)\frac{1}{T(\ln R_0)}+L_1(\ln R)\frac{1}{T(\ln R_1)}+L_2(\ln R)\frac{1}{T(\ln R_2)}+L_3(\ln R)\frac{1}{T(\ln R_3)}$$

substituting $y=\frac{1}{T}$, and $x=\ln R$, the calibration curve is given by

$$y(x)=L_0(x)y(x_0)+L_1(x)y(x_1)+L_2(x)y(x_2)+L_3(x)y(x_3)$$

Find the calibration curve and find the temperature corresponding to 754.8 ohms. What is the difference between the results from cubic interpolation? In which method is the difference larger, if the actual measured value at 754.8 ohms is 35.285 $°C$?

R (Ω)	T (C)	x (ln R)	y (1/T)
1101.0	25.113	7.0040	0.039820
911.3	30.131	6.8149	0.033188
636.0	40.120	6.4552	0.024925
451.1	50.128	6.1117	0.019949

Actual Calibration

$$y(x) = L_0(x)y(x_0) + L_1(x)y(x_1) + L_2(x)y(x_2) + L_3(x)y(x_3)$$

$$x_o = 7.0040, \quad y(x_o) = 0.039820 \qquad x_1 = 6.8149, \quad y(x_1) = 0.033188$$

$$x_2 = 6.4552, \quad y(x_2) = 0.024925 \qquad x_3 = 6.1117, \quad y(x_3) = 0.019949$$

$$L_0(x) = \prod_{\substack{j=0 \\ j\neq 0}}^{3} \frac{x - x_j}{x_0 - x_j} = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)\left(\frac{x - x_3}{x_0 - x_3}\right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j\neq 1}}^{3} \frac{x - x_j}{x_1 - x_j} = \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right)$$

$$L_2(x) = \prod_{\substack{j=0 \\ j\neq 2}}^{3} \frac{x - x_j}{x_2 - x_j} = \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_3}{x_2 - x_3}\right)$$

$$L_3(x) = \prod_{\substack{j=0 \\ j\neq 3}}^{3} \frac{x - x_j}{x_3 - x_j} = \left(\frac{x - x_0}{x_3 - x_0}\right)\left(\frac{x - x_1}{x_3 - x_1}\right)\left(\frac{x - x_2}{x_3 - x_2}\right)$$

Actual Calibration

$$y(x)=\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)\left(\frac{x-x_3}{x_0-x_3}\right)y(x_0)+\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)y(x_1)$$
$$+\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right)y(x_2)+\left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)y(x_3), \quad x_o \le x \le x_3$$

$$x=\ln(754.8)=6.6265$$

$$y(6.6265)=\frac{(6.6265-6.8149)(6.6265-6.4552)(6.6265-6.1117)}{(7.0040-6.8149)(7.0040-6.4552)(7.0040-6.1117)}(0.039820)$$
$$+\frac{(6.6265-7.0040)(6.6265-6.4552)(6.6265-6.1117)}{(6.8149-7.0040)(6.8149-6.4552)(6.8149-6.1117)}(0.033188)$$
$$+\frac{(6.6265-7.0040)(6.6265-6.8149)(6.6265-6.1117)}{(6.4552-7.0040)(6.4552-6.8149)(6.4552-6.1117)}(0.024925)$$
$$+\frac{(6.6265-7.0040)(6.6265-6.8149)(6.6265-6.4552)}{(6.1117-7.0040)(6.1117-6.8149)(6.1117-6.4552)}(0.019949)$$

$$=(-0.17938)(0.039820)+(0.69585)(0.033188)+(0.54005)(0.024925)$$
$$+(-0.056519)(0.019949)$$

$$=0.028285$$

Actual Calibration

Finally, since $y = \frac{1}{T}$, $T = \frac{1}{y} = \frac{1}{0.028285} = 35.355°\text{C}$

Since the actual measured value at 754.8 ohms is 35.285°C, the absolute relative true error for the third order polynomial approximation is

$$|\in_t| = \left|\frac{35.285 - 35.242}{35.285}\right| \times 100$$

$$= 0.12253\%$$

and for the calibration curve used by industry is

$$|\in_t| = \left|\frac{35.285 - 35.355}{35.285}\right| \times 100$$

$$= 0.19825\%$$

Therefore, a cubic polynomial interpolant given by Lagrangian method obtained more accurate results than the calibration curve.

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/lagrange_method.html

THE END

http://numericalmethods.eng.usf.edu