Chapter 05.02
Direct Method of Interpolation

After reading this chapter, you should be able to:

1. *apply the direct method of interpolation,*
2. *solve problems using the direct method of interpolation, and*
3. *use the direct method interpolants to find derivatives and integrals of discrete functions.*

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

(A) evaluate,
(B) differentiate, and
(C) integrate

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ points. One of the methods of interpolation is called the direct method. Other methods include Newton's divided difference polynomial method and the Lagrangian interpolation method. We will discuss the direct method in this chapter.

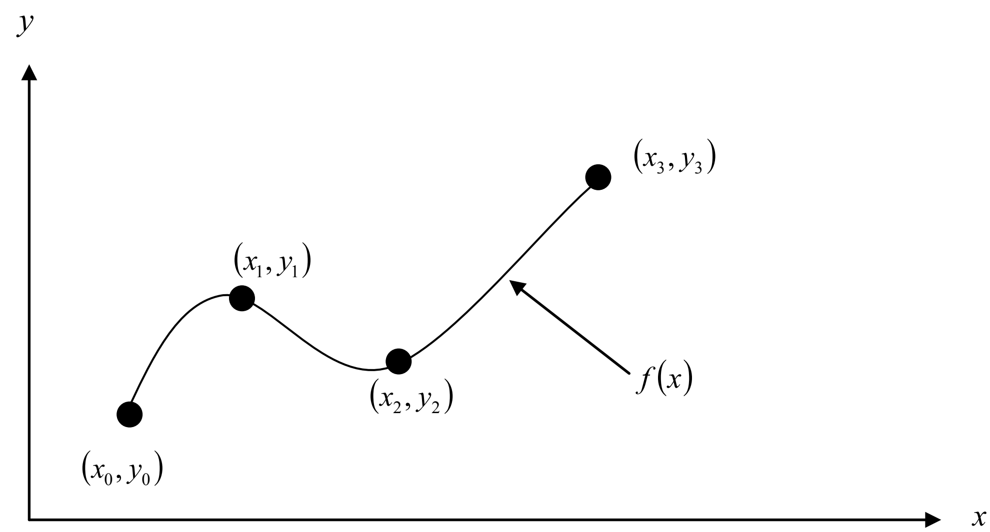

Figure 1 Interpolation of discrete data.

Direct Method

The direct method of interpolation is based on the following premise. Given $n+1$ data points, fit a polynomial of order n as given below

$$y = a_0 + a_1 x + \ldots\ldots\ldots\ldots\ldots + a_n x^n \tag{1}$$

through the data, where $a_0, a_1, \ldots\ldots\ldots, a_n$ are $n+1$ real constants. Since $n+1$ values of y are given at $n+1$ values of x, one can write $n+1$ equations. Then the $n+1$ constants, $a_0, a_1, \ldots\ldots\ldots, a_n$ can be found by solving the $n+1$ simultaneous linear equations. To find the value of y at a given value of x, simply substitute the value of x in Equation 1.

But, it is not necessary to use all the data points. How does one then choose the order of the polynomial and what data points to use? This concept and the direct method of interpolation are best illustrated using examples.

Example 1

Thermistors are used to measure the temperature of bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature vs. resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 1.

Table 1 Temperature as a function of resistance.

R (ohm)	T (°C)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128

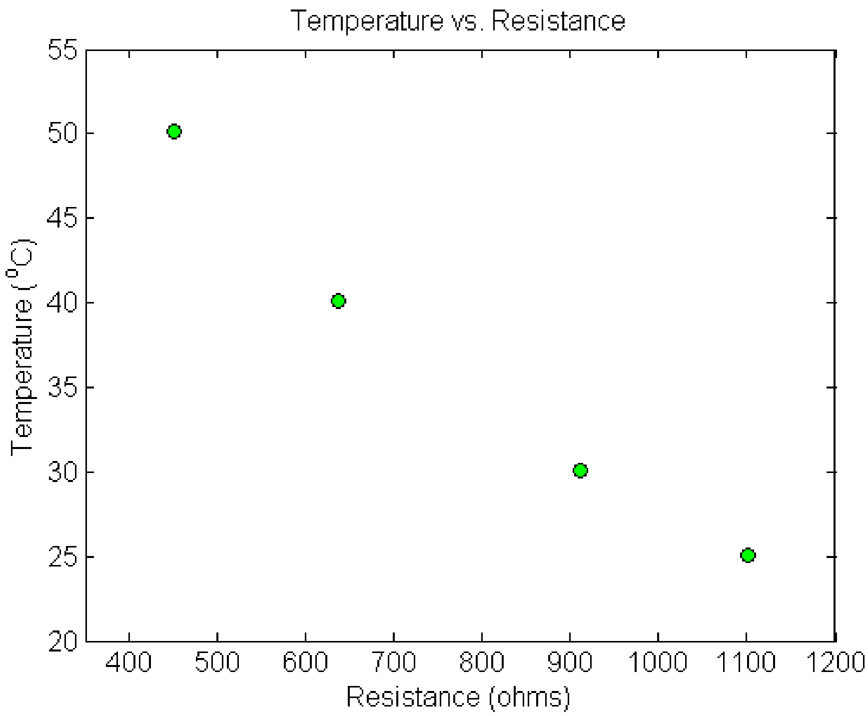

Figure 2 Resistance vs. temperature.

Determine the temperature corresponding to 754.8 ohms using the direct method of interpolation and a first order polynomial.

Solution

For first order polynomial interpolation (also called linear interpolation), we choose the temperature given by

$$T(R) = a_0 + a_1 R$$

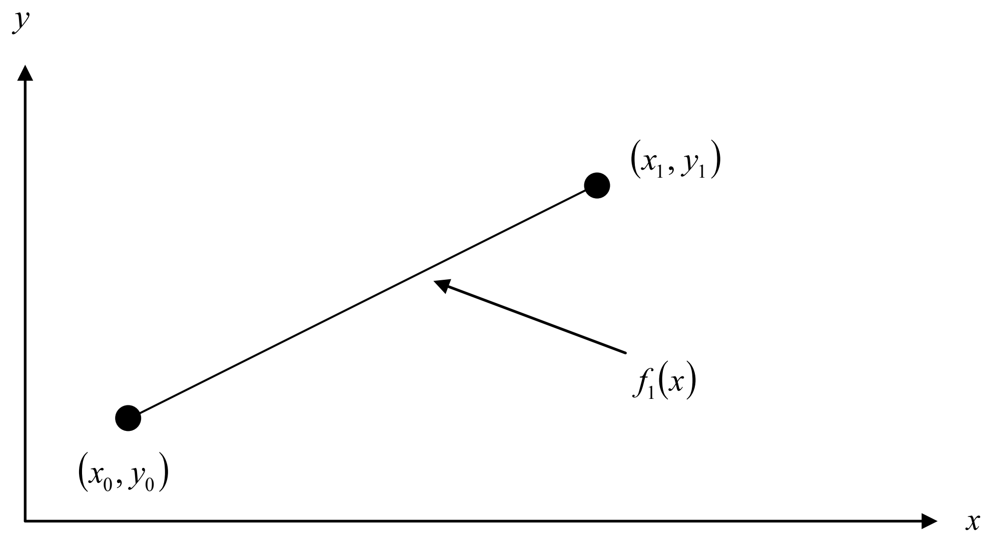

Figure 3 Linear interpolation.

Since we want to find the temperature at $R = 754.8$, and we are using a first order polynomial, we need to choose the two data points that are closest to $R = 754.8$ that also bracket $R = 754.8$ to evaluate it. The two points are $R_0 = 911.3$ and $R_1 = 636.0$.

Then

$$R_0 = 911.3,\ T(R_0) = 30.131$$
$$R_1 = 636.0,\ T(R_1) = 40.120$$

gives

$$T(911.3) = a_0 + a_1(911.3) = 30.131$$
$$T(636.0) = a_0 + a_1(636.0) = 40.120$$

Writing the equations in matrix form, we have

$$\begin{bmatrix} 1 & 911.3 \\ 1 & 636.0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 30.131 \\ 40.120 \end{bmatrix}$$

Solving the above two equations gives

$$a_0 = 63.197$$
$$a_1 = -0.036284$$

Hence

$$T(R) = a_0 + a_1 R$$
$$= 63.197 - 0.036284R, 636.0 \le R \le 911.3$$

At $R = 754.8$,

$$T(754.8) = 63.197 - 0.036284(754.8)$$
$$= 35.809\,°\mathrm{C}$$

Example 2

Thermistors are used to measure the temperature of bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature vs. resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 2.

Table 2 Temperature as a function of resistance.

R (ohm)	T (°C)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128

Determine the temperature corresponding to 754.8 ohms using the direct method of interpolation and a second order polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For second order polynomial interpolation (also called quadratic interpolation), we choose the temperature given by

$$T(R) = a_0 + a_1 R + a_2 R^2$$

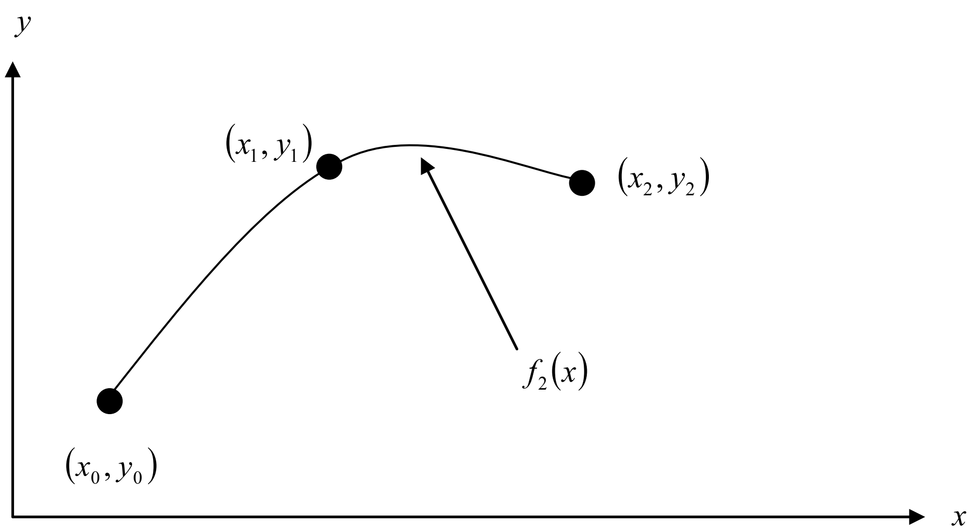

Figure 4 Quadratic interpolation.

Since we want to find the temperature at $R = 754.8$ and we are using a second order polynomial, we need to choose the three data points that are closest to $R = 754.8$ that also bracket $R = 754.8$ to evaluate it. The three points are $R_0 = 911.3$, $R_1 = 636.0$ and $R_2 = 451.1$.

Then

$$R_0 = 911.3,\ T(R_0) = 30.131$$
$$R_1 = 636.0,\ T(R_1) = 40.120$$
$$R_2 = 451.1,\ T(R_2) = 50.128$$

gives

$$T(911.3) = a_0 + a_1(911.3) + a_2(911.3)^2 = 30.131$$
$$T(636.0) = a_0 + a_1(636.0) + a_2(636.0)^2 = 40.120$$
$$T(451.1) = a_0 + a_1(451.1) + a_2(451.1)^2 = 50.128$$

Writing the three equations in matrix form, we have

$$\begin{bmatrix} 1 & 911.3 & 8.3047\times 10^5 \\ 1 & 636.0 & 4.0450\times 10^5 \\ 1 & 451.1 & 2.0349\times 10^5 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 30.131 \\ 40.120 \\ 50.128 \end{bmatrix}$$

Solving the above three equations gives

$$a_0 = 85.668$$
$$a_1 = -0.096275$$
$$a_2 = 3.8771\times 10^{-5}$$

Hence

$$T(R) = 85.668 - 0.096275R + 3.8771\times 10^{-5} R^2,\ \ 451.1 \le R \le 911.3$$

At $R = 754.8$,

$$T(754.8) = 85.668 - 0.096275(754.8) + 3.8771\times 10^{-5}(754.8)^2$$
$$= 35.089\,^\circ\text{C}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{35.089 - 35.809}{35.089}\right| \times 100$$
$$= 2.0543\%$$

Example 3

Thermistors are used to measure the temperature of bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature vs. resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 3.

Table 3 Temperature as a function of resistance.

R (ohm)	T (°C)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128

a) Determine the temperature corresponding to 754.8 ohms using the direct method of interpolation and a third order polynomial. Find the absolute relative approximate error for the third order polynomial approximation.
b) The actual calibration curve used by industry is given by

$$\frac{1}{T} = a_0 + a_1[\ln R] + a_2[\ln R]^2 + a_3[\ln R]^3$$

Substituting $y = \frac{1}{T}$ and $x = \ln R$ the calibration curve is given by

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Table 4 Manipulation for the given data.

R (ohm)	T (°C)	x $(\ln R)$	y $\left(\frac{1}{T}\right)$
1101.0	25.113	7.0040	0.039820
911.3	30.131	6.8149	0.033188
636.0	40.120	6.4552	0.024925
451.1	50.128	6.1117	0.019949

Find the calibration curve and use it to find the temperature corresponding to 754.8 ohms. What is the difference between the results from part (a)? Is the difference larger using results from part (a) or part (b), if the actual measured value at 754.8 ohms is 35.285 °C?

Solution

a) For third order polynomial interpolation (also called cubic interpolation), we choose the temperature given by

$$T(R) = a_0 + a_1 R + a_2 R^2 + a_3 R^3$$

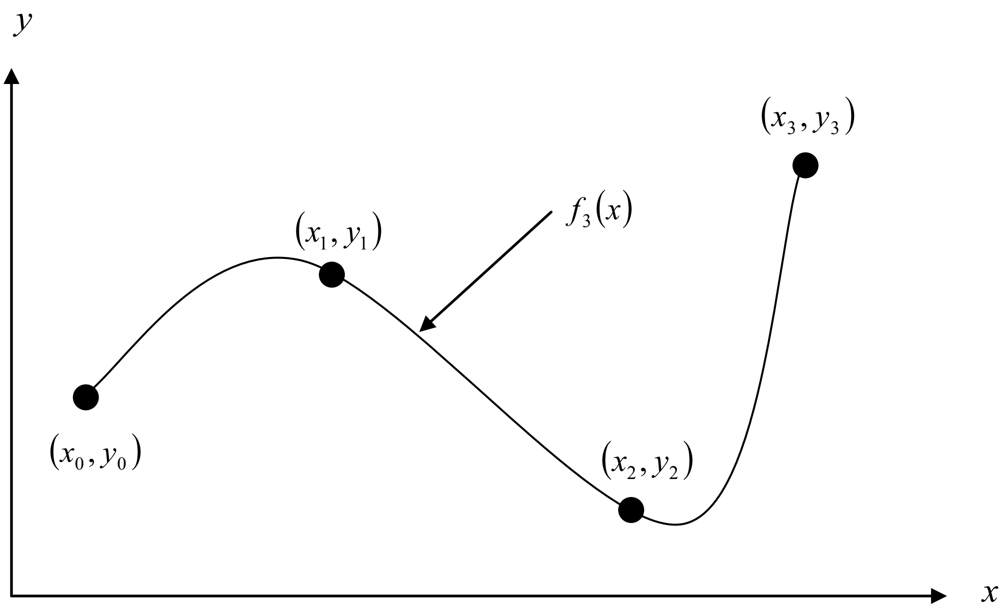

Figure 5 Cubic interpolation.

Since we want to find the temperature at $R = 754.8$, and we are using a third order polynomial, we need to choose the four data points closest to $R = 754.8$ that also bracket $R = 754.8$ to evaluate it. The four points are $R_0 = 1101.0$, $R_1 = 91.3$, $R_2 = 636.0$ and $R_3 = 451.1$.

Then

$$R_0 = 1101.0,\ T(R_0) = 25.113$$
$$R_1 = 911.3,\ T(R_1) = 30.131$$
$$R_2 = 636.0,\ T(R_2) = 40.120$$
$$R_3 = 451.1,\ T(R_3) = 50.128$$

gives

$$T(1101.0) = a_0 + a_1(1101.0) + a_2(1101.0)^2 + a_3(1101.0)^3 = 25.113$$
$$T(911.3) = a_0 + a_1(911.3) + a_2(911.3)^2 + a_3(911.3)^3 = 30.131$$
$$T(636.0) = a_0 + a_1(636.0) + a_2(636.0)^2 + a_3(636.0)^3 = 40.120$$
$$T(451.1) = a_0 + a_1(451.1) + a_2(451.1)^2 + a_3(451.1)^3 = 50.128$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 1101.0 & 1.2122\times10^6 & 1.3346\times10^9 \\ 1 & 911.3 & 8.3047\times10^5 & 7.5681\times10^8 \\ 1 & 636.0 & 4.0450\times10^5 & 2.5726\times10^8 \\ 1 & 451.1 & 2.0349\times10^5 & 9.1795\times10^7 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 25.113 \\ 30.131 \\ 40.120 \\ 50.128 \end{bmatrix}$$

Solving the above four equations gives

$$a_0 = 92.759$$
$$a_1 = -0.13093$$
$$a_2 = 9.2975 \times 10^{-5}$$
$$a_3 = -2.7124 \times 10^{-8}$$

Hence

$$T(R) = a_0 + a_1 R + a_2 R^2 + a_3 R^3$$
$$= 92.759 - 0.13093R + 9.2975 \times 10^{-5} R^2 - 2.7124 \times 10^{-8} R^3,\ 451.1 \le R \le 1101.0$$
$$T(754.8) = 92.759 - 0.13093(754.8) + 9.2975 \times 10^{-5}(754.8)^2 - 2.7124 \times 10^{-8}(754.8)^3$$
$$= 35.242\ ^\circ\mathrm{C}$$

The absolute relative approximate error $|\in_a|$ for the results from the second and third order polynomial is

$$|\in_a| = \left| \frac{35.242 - 35.089}{35.242} \right| \times 100$$
$$= 0.43458\%$$

b) Finding the cubic interpolant using the direct method for

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Requires that we first calculate the new values of x and y.

x $(\ln R)$	y $\left(\frac{1}{T}\right)$
7.0040	0.039820
6.8149	0.033188
6.4552	0.024925
6.1117	0.019949

Then

$$x_0 = 7.0040,\ y(x_0) = 0.039820$$
$$x_1 = 6.8149,\ y(x_1) = 0.033188$$
$$x_2 = 6.4552,\ y(x_2) = 0.024925$$
$$x_3 = 6.1117,\ y(x_3) = 0.019949$$

gives

$$y(7.0040) = a_0 + a_1(7.0040) + a_2(7.0040)^2 + a_3(7.0040)^3 = 0.039820$$
$$y(6.8149) = a_0 + a_1(6.8149) + a_2(6.8149)^2 + a_3(6.8149)^3 = 0.033188$$
$$y(6.4552) = a_0 + a_1(6.4552) + a_2(6.4552)^2 + a_3(6.4552)^3 = 0.024925$$
$$y(6.1117) = a_0 + a_1(6.1117) + a_2(6.1117)^2 + a_3(6.1117)^3 = 0.019949$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 7.0040 & 49.056 & 343.58 \\ 1 & 6.8149 & 46.442 & 316.50 \\ 1 & 6.4552 & 41.670 & 268.99 \\ 1 & 6.1117 & 37.353 & 228.29 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.039820 \\ 0.033188 \\ 0.024925 \\ 0.019949 \end{bmatrix}$$

Solving the above four equations gives

$$a_0 = -2.5964$$
$$a_1 = 1.2605$$
$$a_2 = -0.20448$$
$$a_3 = 0.011173$$

Hence

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$= -2.5964 + 1.2605x - 0.20448x^2 + 0.011173x^3, \quad 6.1117 \le x \le 7.0040$$

However, since $y = \frac{1}{T}$ and $x = \ln R$ we get

$$\frac{1}{T} = -2.5964 + 1.2605(\ln R) - 0.20448(\ln R)^2 + 0.011173(\ln R)^3, \quad 451.1 \le R \le 1101.0$$

or

$$T(R) = \frac{1}{-2.5964 + 1.2605(\ln R) - 0.20448(\ln R)^2 + 0.011173(\ln R)^3}, \quad 451.1 \le R \le 1101.0$$

At $R = 754.8$,

$$T(754.8) = \frac{1}{-2.5964 + 1.2605(\ln(754.8)) - 0.20448(\ln(754.8))^2 + 0.011173(\ln(754.8))^3}$$
$$= 35.355\,^\circ\text{C}$$

Since the actual measured value at 754.8 ohms is $35.285\,^\circ\text{C}$, the absolute relative true error between the value used for part (a) is

$$|\in_t| = \left|\frac{35.285 - 35.242}{35.285}\right| \times 100$$
$$= 0.12253\%$$

and for part (b) is

$$|\in_t| = \left|\frac{35.285 - 35.355}{35.285}\right| \times 100$$
$$= 0.19825\%$$

Therefore, the direct method of cubic polynomial interpolation, that is,

$$T(R) = a_0 + a_1 R + a_2 R^2 + a_3 R^3$$

obtained more accurate results than the actual calibration curve of

$$\frac{1}{T} = a_0 + a_1[\ln R] + a_2[\ln R]^2 + a_3[\ln R]^3$$

INTERPOLATION	
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