Trapezoidal Rule of Integration

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What is Integration

Integration:

The process of measuring the area under a function plotted on a graph.

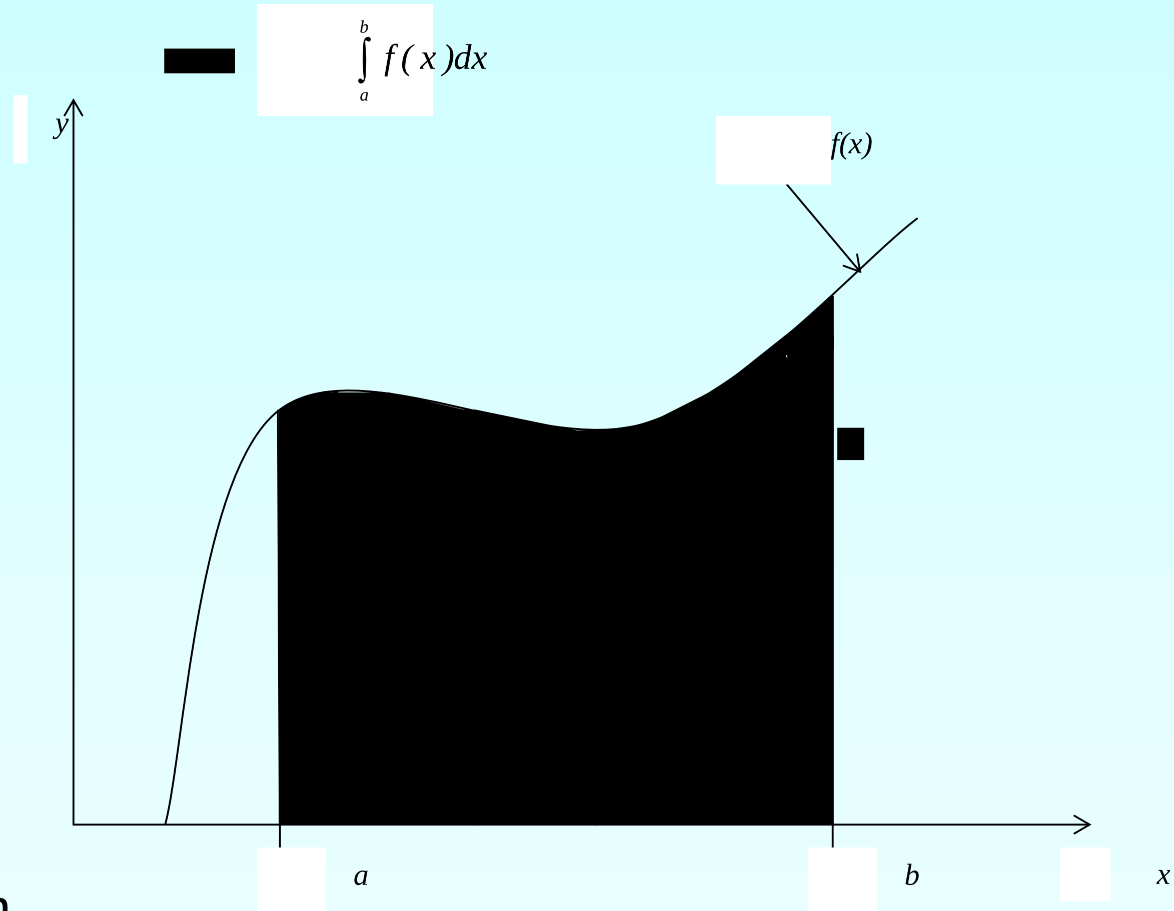
$$I = \int_a^b f(x)dx$$

Where:

f(x) is the integrand

a= lower limit of integration

b= upper limit of integration



Basis of Trapezoidal Rule

Trapezoidal Rule is based on the Newton-Cotes Formula that states if one can approximate the integrand as an n^{th} order polynomial...

$$I = \int_a^b f(x)dx$$ where $$f(x) \approx f_n(x)$$

and $$f_n(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + a_n x^n$$

Basis of Trapezoidal Rule

Then the integral of that function is approximated by the integral of that n^{th} order polynomial.

$$\int_a^b f(x) \approx \int_a^b f_n(x)$$

Trapezoidal Rule assumes n=1, that is, the area under the linear polynomial,

$$\int_a^b f(x)dx \quad = (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

Derivation of the Trapezoidal Rule

Method Derived From Geometry

The area under the curve is a trapezoid. The integral

$$\int_a^b f(x)dx \approx Area\ of\ \ trapezoid$$

$$= \frac{1}{2}(Sum\ of\ \ parallel\ \ sides)(height)$$

$$= \frac{1}{2}\left(f(b) + f(a)\right)(b-a)$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

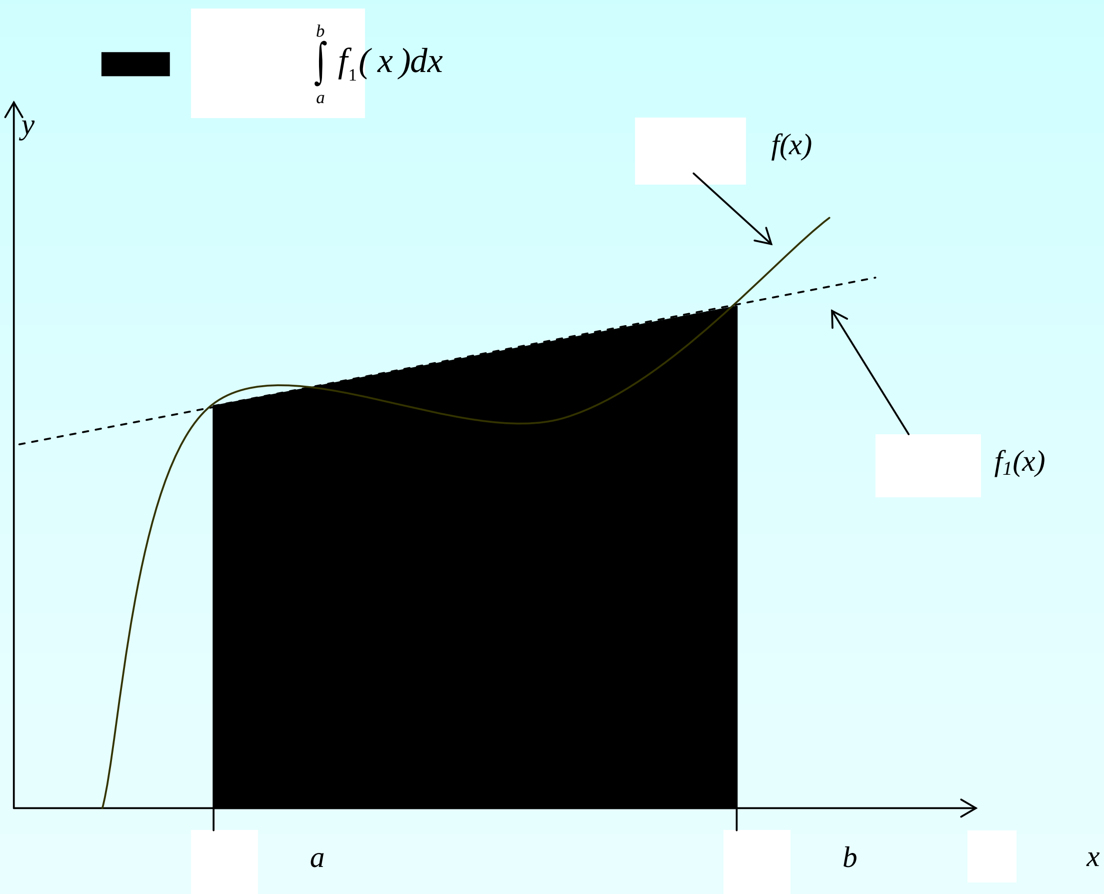


Figure 2: Geometric Representation

Example 1

The probability for an oscillator to have its frequency within 5% of the target of 1kHz is determined by finding total area under the normal distribution function for the range in question:

$$(1-\alpha) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

a) Use single segment Trapezoidal rule to find the frequency
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $|\in_a|$ for part (a).

Solution

a) $$I \approx (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

$$a = -2.15 \qquad b = 2.9$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$f(-2.15) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(-2.15)^2}{2}} = 0.03955$$

$$f(2.9) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(2.9)^2}{2}} = 0.0059525$$

Solution (cont)

a)

$$I = \left(2.9 - \left(-2.15\right)\right)\left[\frac{0.03955 + 0.0059525}{2}\right]$$

$$= 0.11489$$

b) Since the exact value of the above integral cannot be found, we take numerical integration value using maple as exact value

$$\left(1-\alpha\right) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0.98236$$

Solution (cont)

b) $E_t = True\ Value - Approximate\ Value$

$$= 0.98236 - 0.11489$$

$$= 0.86746$$

c) The absolute relative true error, $|\in_t|$, would be

$$|\in_t| = \left|\frac{0.98236 - 0.11489}{0.98236}\right| \times 100 = 88.304\%$$

Multiple Segment Trapezoidal Rule

In Example 1, the true error using single segment trapezoidal rule was large. We can divide the interval [8,30] into [8,19] and [19,30] intervals and apply Trapezoidal rule over each segment.

$$f(t) = 2000\,ln\left(\frac{140000}{140000 - 2100t}\right) - 9.8t$$

$$\int_{8}^{30} f(t)dt = \int_{8}^{19} f(t)dt + \int_{19}^{30} f(t)dt$$

$$= (19-8)\left[\frac{f(8) + f(19)}{2}\right] + (30-19)\left[\frac{f(19) + f(30)}{2}\right]$$

Multiple Segment Trapezoidal Rule

With

$$f(8) = 177.27\ m/s$$

$$f(30) = 901.67\ m/s$$

$$f(19) = 484.75\ m/s$$

Hence:

$$\int_8^{30} f(t)dt = (19-8)\left[\frac{177.27+484.75}{2}\right] + (30-19)\left[\frac{484.75+901.67}{2}\right]$$

$$= 11266\ m$$

Multiple Segment Trapezoidal Rule

The true error is:

$$E_t = 11061 - 11266$$

$$= -205\ m$$

The true error now is reduced from -807 m to -205 m.

Extending this procedure to divide the interval into equal segments to apply the Trapezoidal rule; the sum of the results obtained for each segment is the approximate value of the integral.

Multiple Segment Trapezoidal Rule

Divide into equal segments as shown in Figure 4. Then the width of each segment is:

$$h = \frac{b-a}{n}$$

The integral I is:

$$I = \int_a^b f(x)dx$$

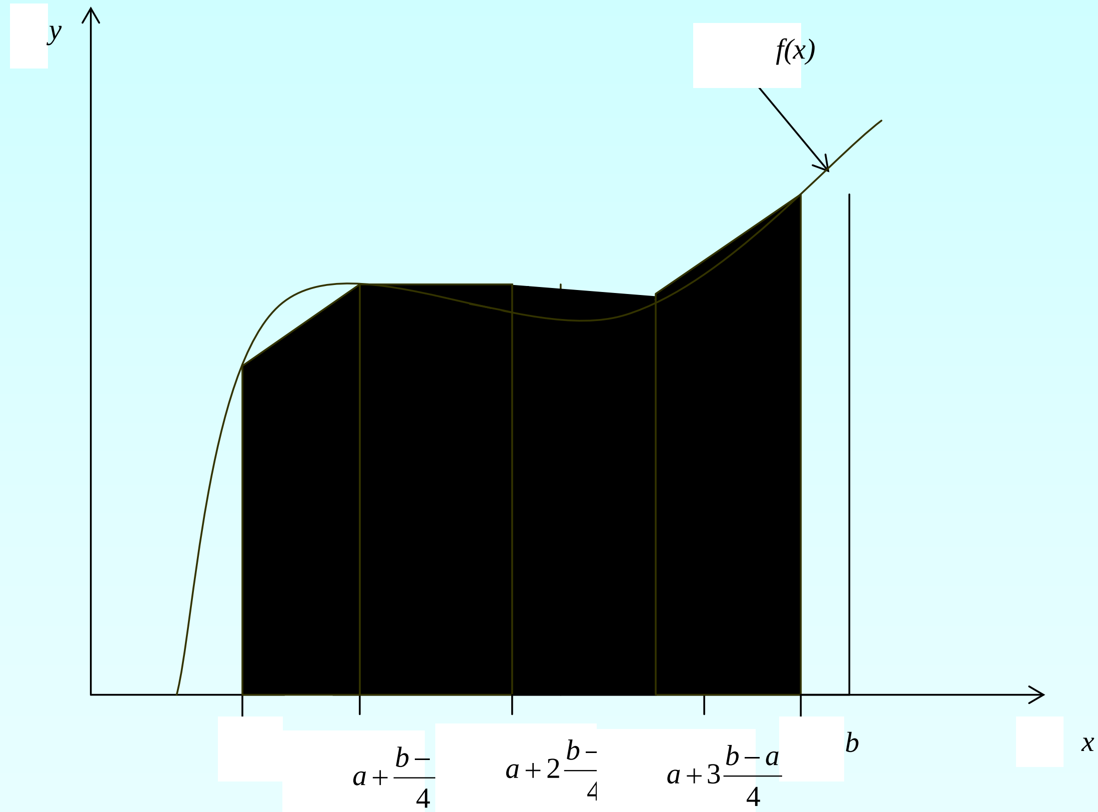


Figure 4: Multiple (n=4) Segment Trapezoidal Rule

Multiple Segment Trapezoidal Rule

The integral *I* can be broken into *h* integrals as:

$$\int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \ldots + \int_{a+(n-2)h}^{a+(n-1)h} f(x)dx + \int_{a+(n-1)h}^{b} f(x)dx$$

Applying Trapezoidal rule on each segment gives:

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a) + 2\left\{ \sum_{i=1}^{n-1} f(a+ih) \right\} + f(b) \right]$$

Example 2

The probability for an oscillator to have its frequency within 5% of the target of 1kHz is determined by finding total area under the normal distribution function for the range in question:

$$(1-\alpha) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

a) Use two-segment Trapezoidal rule to find the frequency.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $|\in_a|$ for part (a).

Solution

a) The solution using 2-segment Trapezoidal rule is

$$I \approx \frac{b-a}{2n}\left[f(a)+2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\}+f(b)\right]$$

$$n = 2 \qquad a = -2.15 \qquad b = 2.9$$

$$h = \frac{b-a}{n} = \frac{2.9-(-2.15)}{2}$$

Solution (cont)

Then:

$$I \approx \frac{2.9-(-2.15)}{2(2)}\left[f(-2.15)+2\left\{\sum_{i=1}^{2-1} f(a+ih)\right\}+ f(2.9)\right]$$

$$\approx \frac{5.05}{4}\left[f\left(-2.15\right)+2f\left(0.375\right)+ f\left(2.9\right)\right]$$

$$\approx \frac{5.05}{4}\left[0.039550+2\left(0.37186\right)+0.0059525\right]$$

$$\approx 0.99638$$

Solution (cont)

Since the exact value of the above integral cannot be found, we take numerical integration value using maple as exact value

$$(1-\alpha)=\int_{-2.15}^{2.9}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx=0.98236$$

so the true error is

$$\begin{aligned}E_t &= True\ Value - Approximate\ Value\\ &= 0.98236-0.99638\\ &= -0.014025\end{aligned}$$

Solution (cont)

c) The absolute relative true error, $\left|\in_t\right|$, would be

$$\left|\in_t\right| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{0.98236 - 0.99638}{0.98236}\right| \times 100$$

$$= 1.4276\%$$

Solution (cont)

Table 1 gives the values obtained using multiple segment Trapezoidal rule for:

$$(1-\alpha)=\int_{-2.15}^{2.9}\frac{1}{\sqrt{2\pi}}e^{\frac{x^2}{2}}dx$$

n	Value	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
1	0.11489	0.86746	88.304	---
2	0.99638	−0.014025	1.4276	88.469
3	0.96093	0.021427	2.1812	3.6891
4	0.96969	0.012670	1.2897	0.90338
5	0.97402	0.0083332	0.84829	0.44455
6	0.97649	0.0058680	0.59734	0.25259
7	0.97801	0.0043459	0.44239	0.15542
8	0.97901	0.0033441	0.34042	0.10214

Table 1: Multiple Segment Trapezoidal Rule Values

Example 3

Use Multiple Segment Trapezoidal Rule to find the area under the curve

$$f(x)=\frac{300x}{1+e^{x}} \quad \text{from} \quad x=0 \quad \text{to} \quad x=10$$

Using two segments, we get $h=\frac{10-0}{2}=5$ and

$$f(0)=\frac{300(0)}{1+e^{0}}=0 \qquad f(5)=\frac{300(5)}{1+e^{5}}=10.039 \qquad f(10)=\frac{300(10)}{1+e^{10}}=0.136$$

Solution

Then:

$$I = \frac{b-a}{2n}\left[f(a) + 2\left\{ \sum_{i=1}^{n-1} f(a+ih) \right\} + f(b) \right]$$

$$= \frac{10-0}{2(2)}\left[f(0) + 2\left\{ \sum_{i=1}^{2-1} f(0+5) \right\} + f(10) \right]$$

$$= \frac{10}{4}\left[f(0) + 2f(5) + f(10) \right] = \frac{10}{4}\left[0 + 2(10.039) + 0.136 \right]$$

$$= 50.535$$

Solution (cont)

So what is the true value of this integral?

$$\int_0^{10} \frac{300x}{1+e^x}\,dx = 246.59$$

Making the absolute relative true error:

$$\left|\in_t\right| = \left|\frac{246.59 - 50.535}{246.59}\right| \times 100\%$$

$$= 79.506\%$$

Solution (cont)

Table 2: Values obtained using Multiple Segment Trapezoidal Rule for:

$$\int_0^{10} \frac{300x}{1+e^x}\,dx$$

n	Approximate Value	E_t	$\lvert\in_t\rvert$
1	0.681	245.91	99.724%
2	50.535	196.05	79.505%
4	170.61	75.978	30.812%
8	227.04	19.546	7.927%
16	241.70	4.887	1.982%
32	245.37	1.222	0.495%
64	246.28	0.305	0.124%

Error in Multiple Segment Trapezoidal Rule

The true error for a single segment Trapezoidal rule is given by:

$$E_t = \frac{(b-a)^3}{12} f''(\zeta), \quad a < \zeta < b$$ where ζ is some point in $[a,b]$

What is the error, then in the multiple segment Trapezoidal rule? It will be simply the sum of the errors from each segment, where the error in each segment is that of the single segment Trapezoidal rule.

The error in each segment is

$$E_1 = \frac{[(a+h)-a]^3}{12} f''(\zeta_1), \quad a < \zeta_1 < a+h$$

$$= \frac{h^3}{12} f''(\zeta_1)$$

Error in Multiple Segment Trapezoidal Rule

Similarly:

$$E_i = \frac{\left[(a+ih)-(a+(i-1)h)\right]^3}{12} f''(\zeta_i), \quad a+(i-1)h < \zeta_i < a+ih$$

$$= \frac{h^3}{12} f''(\zeta_i)$$

It then follows that:

$$E_n = \frac{\left[b-\{a+(n-1)h\}\right]^3}{12} f''(\zeta_n), \quad a+(n-1)h < \zeta_n < b$$

$$= \frac{h^3}{12} f''(\zeta_n)$$

Error in Multiple Segment Trapezoidal Rule

Hence the total error in multiple segment Trapezoidal rule is

$$E_t = \sum_{i=1}^{n} E_i \quad = \frac{h^3}{12} \sum_{i=1}^{n} f''(\zeta_i) \quad = \frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

The term $\dfrac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$ is an approximate average value of the $f''(x),\ a < x < b$

Hence:

$$E_t = \frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

Error in Multiple Segment Trapezoidal Rule

Below is the table for the integral $\int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$

as a function of the number of segments. You can visualize that as the number of segments are doubled, the true error gets approximately quartered.

n	**Value**	E_t	$\lvert \in_t \rvert \%$	$\lvert \in_a \rvert \%$
2	11266	-205	1.854	5.343
4	11113	-51.5	0.4655	0.3594
8	11074	-12.9	0.1165	0.03560
16	11065	-3.22	0.02913	0.00401

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/trapezoidal_rule.html

THE END

http://numericalmethods.eng.usf.edu