Chapter 08.02
Euler's Method for Ordinary Differential Equations

After reading this chapter, you should be able to:

1. *develop Euler's Method for solving ordinary differential equations,*
2. *determine how the step size affects the accuracy of a solution,*
3. *derive Euler's formula from Taylor series, and*
4. *use Euler's method to find approximate values of integrals.*

What is Euler's method?

Euler's method is a numerical technique to solve ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0 \tag{1}$$

So only first order ordinary differential equations can be solved by using Euler's method. In another chapter we will discuss how Euler's method is used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations. How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y),\ y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x,y)=1.3e^{-x}-2y$$

Example 2

Rewrite

$$e^{y}\frac{dy}{dx}+x^{2}y^{2}=2\sin(3x),\ y(0)=5$$

in

$$\frac{dy}{dx}=f(x,y),\ y(0)=y_0 \text{ form.}$$

Solution

$$e^{y}\frac{dy}{dx}+x^{2}y^{2}=2\sin(3x),\ y(0)=5$$

$$\frac{dy}{dx}=\frac{2\sin(3x)-x^{2}y^{2}}{e^{y}},\ y(0)=5$$

In this case

$$f(x,y)=\frac{2\sin(3x)-x^{2}y^{2}}{e^{y}}$$

Derivation of Euler's method

At $x=0$, we are given the value of $y=y_0$. Let us call $x=0$ as x_0. Now since we know the slope of y with respect to x, that is, $f(x,y)$, then at $x=x_0$, the slope is $f(x_0,y_0)$. Both x_0 and y_0 are known from the initial condition $y(x_0)=y_0$.

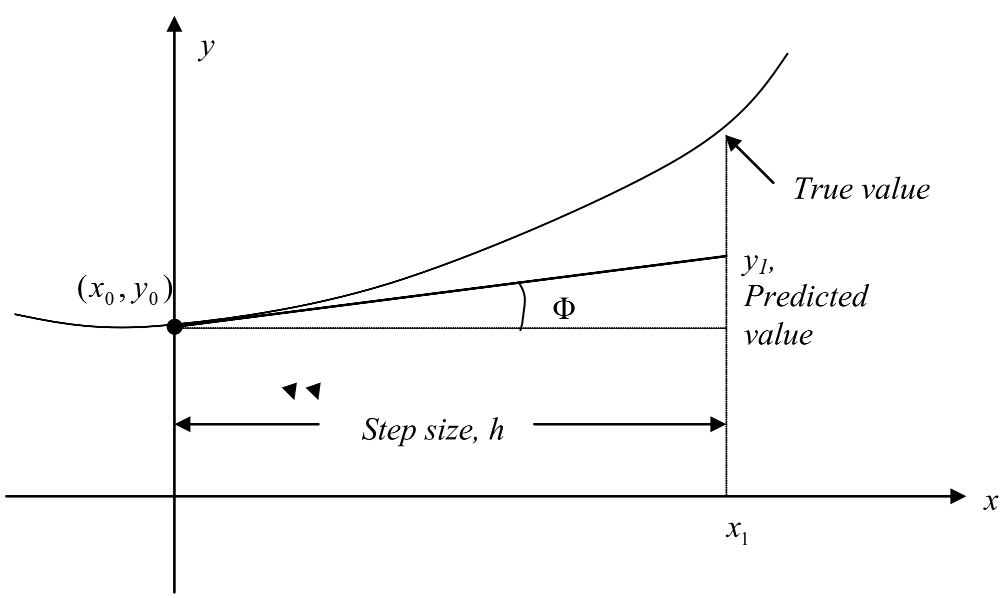

Figure 1 Graphical interpretation of the first step of Euler's method.

So the slope at $x = x_0$ as shown in Figure 1 is

$$\text{Slope} = \frac{Rise}{Run}$$
$$= \frac{y_1 - y_0}{x_1 - x_0}$$
$$= f(x_0, y_0)$$

From here

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

Calling $x_1 - x_0$ the step size h, we get

$$y_1 = y_0 + f(x_0, y_0)h \tag{2}$$

One can now use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2, and that would be the predicted value at x_2, given by

$$y_2 = y_1 + f(x_1, y_1)h$$
$$x_2 = x_1 + h$$

Based on the above equations, if we now know the value of $y = y_i$ at x_i, then

$$y_{i+1} = y_i + f(x_i, y_i)h \tag{3}$$

This formula is known as Euler's method and is illustrated graphically in Figure 2. In some books, it is also called the Euler-Cauchy method.

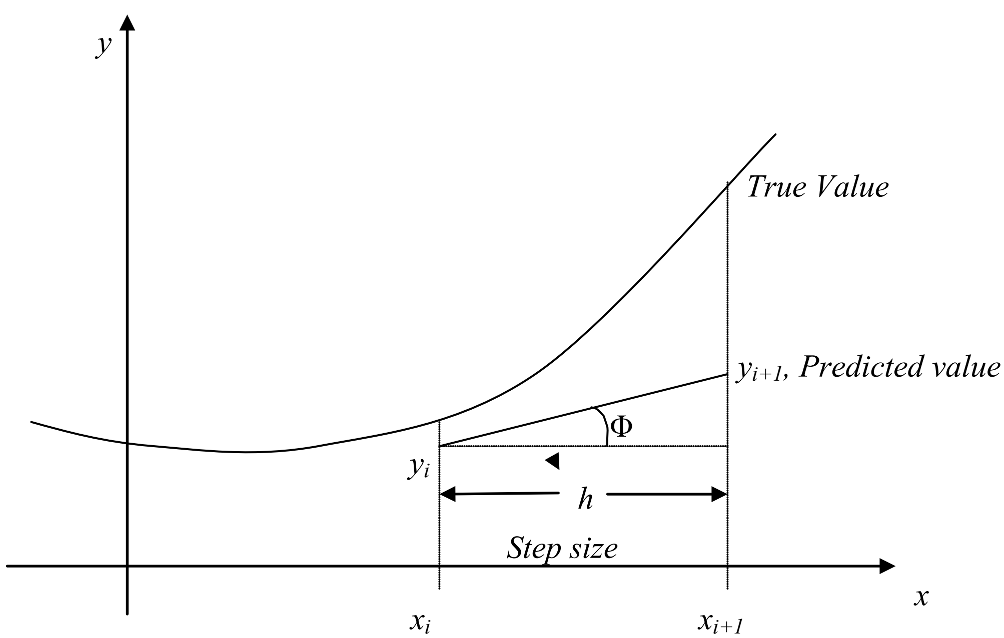

Figure 2 General graphical interpretation of Euler's method.

Example 3

A rectifier-based power supply requires a capacitor to temporarily store power when the rectified waveform from the AC source drops below the target voltage. To properly size this capacitor a first-order ordinary differential equation must be solved. For a particular power supply, with a capacitor of $150\ \mu\text{F}$, the ordinary differential equation to be solved is

$$\frac{dv(t)}{dt}=\frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(t))|-2-v(t)}{0.04},0\right)\right\}$$

$$v(0)=0$$

Using Euler's method, find the voltage across the capacitor at $t=0.00004\,\text{s}$. Use step size $h=0.00002\,\text{s}$.

Solution

$$\frac{dv}{dt}=\frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(t))|-2-v}{0.04},0\right)\right\}$$

$$f(t,v)=\frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(t))|-2-v}{0.04},0\right)\right\}$$

The Euler's method reduces to

$$v_{i+1}=v_i+f(t_i,v_i)h$$

For $i=0$, $t_0=0$, $v_0=0$

$$\begin{aligned}v_1&=v_0+f(t_0,v_0)h\\&=0+f(0,0)0.00002\\&=\frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(0))|-2-(0)}{0.04},0\right)\right\}0.00002\\&=0+(2.666\times10^6)0.00002\\&=53.320\,\text{V}\end{aligned}$$

v_1 is the approximate voltage at

$$t=t_1=t_0+h=0+0.00002=0.00002\,\text{s}$$

$$v(0.00002)\approx v_1=53.320\,\text{V}$$

For $i=1$, $t_1=0.00002$, $v_1=53.320$

$$\begin{aligned}v_2&=v_1+f(t_1,v_1)h\\&=53.320+f(0.00002,53.320)0.00002\\&=53.320+\frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(0.00002))|-2-(53.320)}{0.04},0\right)\right\}0.00002\end{aligned}$$

$$= 53.320 + (-0.000015000)0.00002$$
$$= 53.307\ \text{V}$$

v_2 is the approximate voltage at

$$t = t_2 = t_1 + h = 0.00002 + 0.00002 = 0.00004\ \text{s}$$
$$v(0.00004) \approx v_2 = 53.307\ \text{V}$$

Figure 3 compares the exact solution of $v(0.00004) = 15.974\ \text{V}$ with the numerical solution from Euler's method for the step size of $h = 0.00004\ \text{s}$.

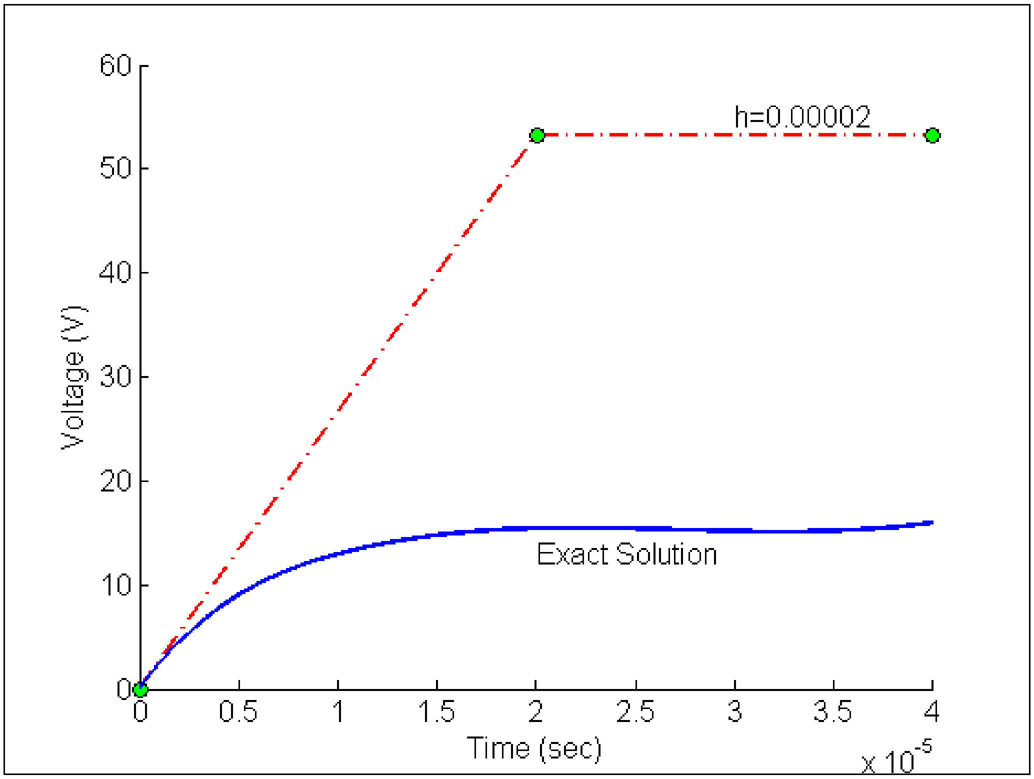

Figure 3 Comparing exact and Euler's method.

The problem was solved again using smaller step sizes. The results are given below in Table 1.

Table 1 Voltage at 0.00004 seconds as a function of step size, h.

Step size, h	$v(0.00004)$	E_t	$\lvert\in_t\rvert\ \%$
0.00004	106.64	−90.667	567.59
0.00002	53.307	−37.333	233.71
0.00001	26.640	−10.666	66.771
0.000005	15.996	−0.021991	0.13766
0.0000025	15.993	−0.019125	0.11972

Figure 4 shows how the voltage varies as a function of time for different step sizes.

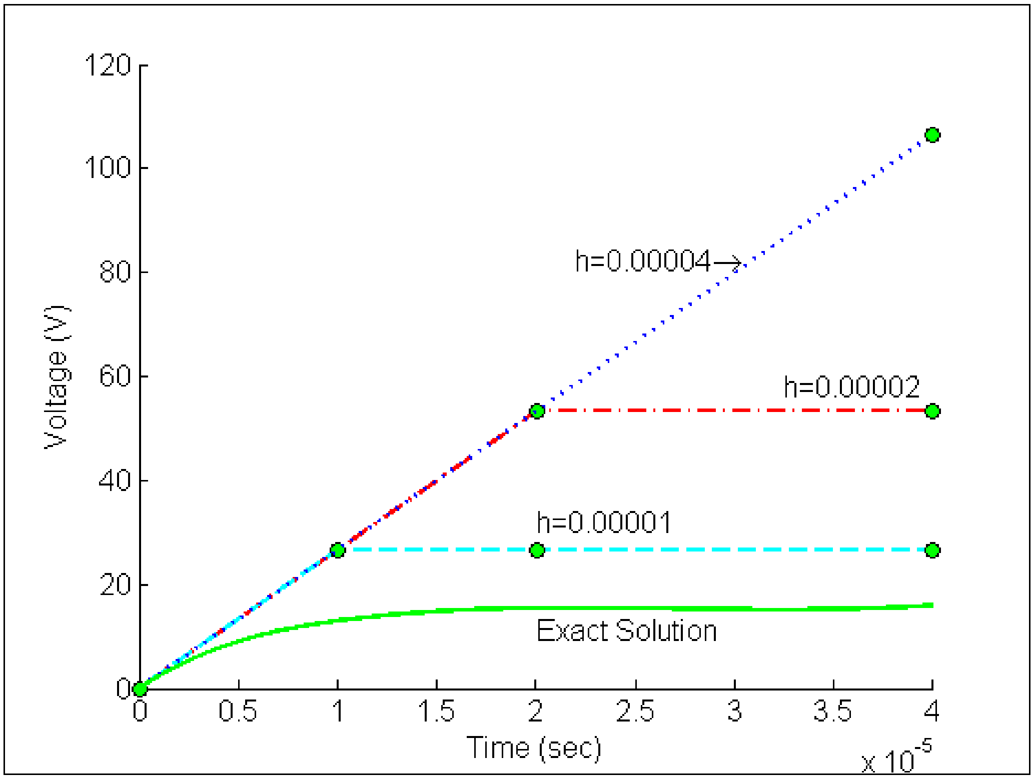

Figure 4 Comparison of Euler's method with exact solution for different step sizes.

While the values of the calculated voltage at $t = 0.00004\,\text{s}$ as a function of step size are plotted in Figure 5.

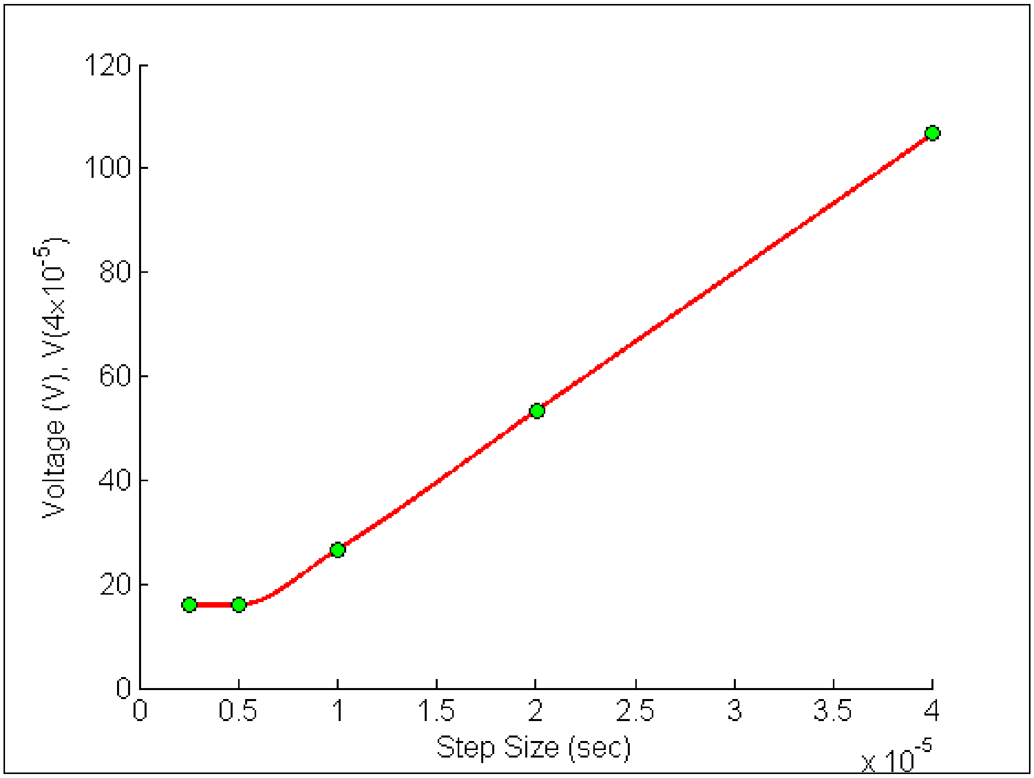

Figure 5 Effect of step size in Euler's method.

Can one solve a definite integral using numerical methods such as Euler's method of solving ordinary differential equations?

Let us suppose you want to find the integral of a function $f(x)$

$$I = \int_a^b f(x)dx .$$

Both fundamental theorems of calculus would be used to set up the problem so as to solve it as an ordinary differential equation.

The first fundamental theorem of calculus states that if f is a continuous function in the interval [a,b], and F is the antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

The second fundamental theorem of calculus states that if f is a continuous function in the open interval D, and a is a point in the interval D, and if

$$F(x) = \int_a^x f(t)dt$$

then

$$F'(x) = f(x)$$

at each point in D.

Asked to find $\int_a^b f(x)dx$, we can rewrite the integral as the solution of an ordinary differential equation (here is where we are using the second fundamental theorem of calculus)

$$\frac{dy}{dx} = f(x),\ y(a) = 0,$$

where then $y(b)$ (here is where we are using the first fundamental theorem of calculus) will give the value of the integral $\int_a^b f(x)dx$.

Example 4

Find an approximate value of

$$\int_5^8 6x^3 dx$$

using Euler's method of solving an ordinary differential equation. Use a step size of $h = 1.5$.

Solution

Given $\int_5^8 6x^3 dx$, we can rewrite the integral as the solution of an ordinary differential equation

$$\frac{dy}{dx} = 6x^3,\ y(5) = 0$$

where $y(8)$ will give the value of the integral $\int_5^8 6x^3 dx$.

$$\frac{dy}{dx} = 6x^3 = f(x,y),\ y(5) = 0$$

The Euler's method equation is

$$y_{i+1} = y_i + f(x_i, y_i)h$$

<u>Step 1</u>

$$i = 0,\ x_0 = 5,\ y_0 = 0$$
$$h = 1.5$$
$$\begin{aligned} x_1 &= x_0 + h \\ &= 5 + 1.5 \\ &= 6.5 \end{aligned}$$
$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)h \\ &= 0 + f(5,0) \times 1.5 \\ &= 0 + (6 \times 5^3) \times 1.5 \\ &= 1125 \\ &\approx y(6.5) \end{aligned}$$

Step 2

$$i = 1,\ x_1 = 6.5,\ y_1 = 1125$$

$$\begin{aligned} x_2 &= x_1 + h \\ &= 6.5 + 1.5 \\ &= 8 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + f(x_1, y_1)h \\ &= 1125 + f(6.5, 1125) \times 1.5 \\ &= 1125 + \left(6 \times 6.5^3\right) \times 1.5 \\ &= 3596.625 \\ &\approx y(8) \end{aligned}$$

Hence

$$\begin{aligned} \int_5^8 6x^3 dx &= y(8) - y(5) \\ &\approx 3596.625 - 0 \\ &= 3596.625 \end{aligned}$$

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Euler's Method for ordinary differential equations
Summary	Textbook notes on Euler's method for solving ordinary differential equations
Major	Electrical Engineering
Authors	Autar Kaw
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Web Site	http://numericalmethods.eng.usf.edu