Propagation of Errors

Major: All Engineering Majors

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Numerical Methods for STEM undergraduates

Propagation of Errors

In numerical methods, the calculations are not made with exact numbers. How do these inaccuracies propagate through the calculations?

Example 1:

Find the bounds for the propagation in adding two numbers. For example if one is calculating $X+Y$ where

$$X = 1.5 \pm 0.05$$
$$Y = 3.4 \pm 0.04$$

Solution

Maximum possible value of $X = 1.55$ and $Y = 3.44$

Maximum possible value of $X + Y = 1.55 + 3.44 = 4.99$

Minimum possible value of $X = 1.45$ and $Y = 3.36$.

Minimum possible value of $X + Y = 1.45 + 3.36 = 4.81$

Hence

$$4.81 \leq X + Y \leq 4.99.$$

Propagation of Errors In Formulas

If f is a function of several variables $X_1, X_2, X_3, \ldots\ldots., X_{n-1}, X_n$ then the maximum possible value of the error in f is

$$\Delta f \approx \left|\frac{\partial f}{\partial X_1}\Delta X_1\right| + \left|\frac{\partial f}{\partial X_2}\Delta X_2\right| + \ldots\ldots. + \left|\frac{\partial f}{\partial X_{n-1}}\Delta X_{n-1}\right| + \left|\frac{\partial f}{\partial X_n}\Delta X_n\right|$$

Example 2:

The strain in an axial member of a square cross-section is given by

$$\in=\frac{F}{h^2E}$$

Given

$$F=72\pm0.9\ \text{N}$$
$$h=4\pm0.1\ \text{mm}$$
$$E=70\pm1.5\ \text{GPa}$$

Find the maximum possible error in the measured strain.

Example 2:

Solution

$$\in = \frac{72}{(4\times 10^{-3})^2 (70\times 10^9)}$$

$$= 64.286\times 10^{-6}$$

$$= 64.286\mu$$

$$\Delta \in = \left|\frac{\partial \in}{\partial F}\Delta F\right| + \left|\frac{\partial \in}{\partial h}\Delta h\right| + \left|\frac{\partial \in}{\partial E}\Delta E\right|$$

Example 2:

$$\frac{\partial \in}{\partial F} = \frac{1}{h^2 E} \qquad \frac{\partial \in}{\partial h} = -\frac{2F}{h^3 E} \qquad \frac{\partial \in}{\partial E} = -\frac{F}{h^2 E^2}$$

Thus

$$\Delta E = \left|\frac{1}{h^2 E}\Delta F\right| + \left|\frac{2F}{h^3 E}\Delta h\right| + \left|\frac{F}{h^2 E^2}\Delta E\right|$$

$$= \left|\frac{1}{(4\times10^{-3})^2(70\times10^9)}\times0.9\right| + \left|\frac{2\times72}{(4\times10^{-3})^3(70\times10^9)}\times0.0001\right|$$

$$+ \left|\frac{72}{(4\times10^{-3})^2(70\times10^9)^2}\times1.5\times10^9\right|$$

$$= 5.3955\mu$$

Hence

$$\in = (64.286\mu \pm 5.3955\mu)$$

Example 3:

Subtraction of numbers that are nearly equal can create unwanted inaccuracies. Using the formula for error propagation, show that this is true.

Solution

Let

$$z = x - y$$

Then

$$|\Delta z| = \left|\frac{\partial z}{\partial x}\Delta x\right| + \left|\frac{\partial z}{\partial y}\Delta y\right|$$

$$= |(1)\Delta x| + |(-1)\Delta y|$$

$$= |\Delta x| + |\Delta y|$$

So the relative change is

$$\left|\frac{\Delta z}{z}\right| = \frac{|\Delta x| + |\Delta y|}{|x - y|}$$

Example 3:

For example if

$$x = 2 \pm 0.001$$

$$y = 2.003 \pm 0.001$$

$$\left|\frac{\Delta z}{z}\right| = \frac{|0.001| + |0.001|}{|\,2 - 2.003\,|}$$

$$= 0.6667$$

$$= 66.67\%$$