Chapter 02.03
Differentiation of Discrete Functions

After reading this chapter, you should be able to:

1. *find approximate values of the first derivative of functions that are given at discrete data points, and*
2. *use Lagrange polynomial interpolation to find derivatives of discrete functions.*

To find the derivatives of functions that are given at discrete points, several methods are available. Although these methods are mainly used when the data is spaced unequally, they can be used for data that is spaced equally as well.

Forward Difference Approximation of the First Derivative

We know

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

For a finite Δx,

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

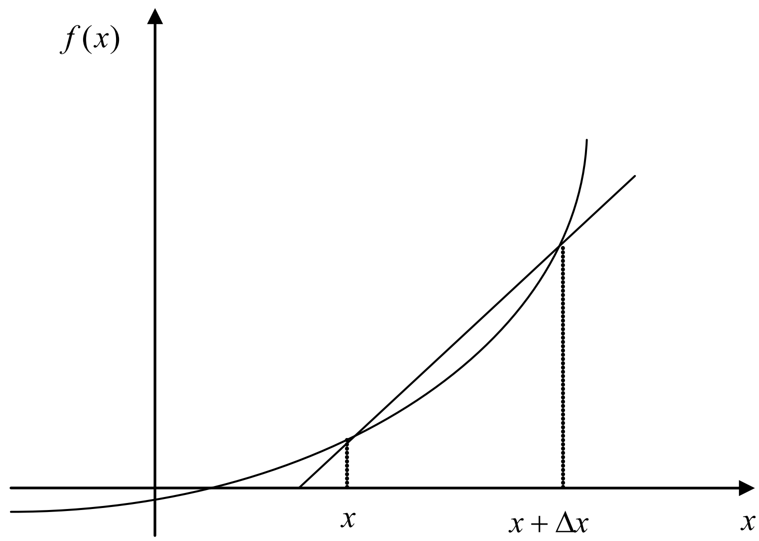

Figure 1 Graphical representation of forward difference approximation of first derivative.

So given $n+1$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, the value of $f'(x)$ for $x_i \le x \le x_{i+1}$, $i = 0,\ldots, n-1$, is given by

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

Example 1

The failure rate $h(t)$ of a direct methanol fuel cell (DMFC) is given by the formula

$$h(t) = -\frac{R'(t)}{R(t)}$$

where $R(t)$ is the reliability at a certain time t, and the values of the reliability are given in Table 1.

Table 1 Reliability of DMFC system.

t (hrs)	0	1	10	100	1000	2000	3000	4000	5000
$R(t)$	1	0.9999	0.9998	0.9980	0.9802	0.9609	0.9419	0.9233	0.9050

Using the forward divided difference method, find the failure rate of the DMFC system at $t = 50$ hours.

Solution

$$R'(t_i) \approx \frac{R(t_{i+1}) - R(t_i)}{\Delta t}$$

$$t_i = 10$$

$$t_{i+1} = 100$$
$$\Delta t = t_{i+1} - t_i$$
$$= 100 - 10$$
$$= 90$$
$$R'(50) \approx \frac{R(100) - R(10)}{90}$$
$$= \frac{0.9980 - 0.9998}{90}$$
$$= -2.0000 \times 10^{-5}$$

The reliability $R(t)$ at $t = 50$ hours is,

$$R(50) \approx \frac{R(100) - R(10)}{100 - 10}(50 - 10) + R(10)$$
$$= (-2.0000 \times 10^{-5})(40) + 0.9998$$
$$= 0.999$$

The failure rate $h(t)$ at $t = 50$ hours is then,

$$h(50) = -\frac{R'(50)}{R(50)}$$
$$h(50) = -\frac{(-2.0000 \times 10^{-5})}{0.999}$$
$$h(50) = 2.0020 \times 10^{-5}$$

Direct Fit Polynomials

In this method, given $n+1$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, one can fit a n^{th} order polynomial given by

$$P_n(x) = a_0 + a_1 x + \ldots\ldots + a_{n-1} x^{n-1} + a_n x^n$$

To find the first derivative,

$$P_n'(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2 x + \ldots\ldots + (n-1)a_{n-1} x^{n-2} + n a_n x^{n-1}$$

Similarly, other derivatives can also be found.

Example 2

The failure rate $h(t)$ of a direct methanol fuel cell (DMFC) is given by the formula

$$h(t) = -\frac{R'(t)}{R(t)}$$

where $R(t)$ is the reliability at a certain time t, and the values of the reliability are given in Table 2.

Table 2 Reliability of DMFC system.

t (hrs)	0	1	10	100	1000	2000	3000	4000	5000
$R(t)$	1	0.9999	0.9998	0.9980	0.9802	0.9609	0.9419	0.9233	0.9050

Using a third order polynomial interpolant for reliability $R(t)$, find the failure rate of the DMFC at $t = 50$ hours.

Solution

For third order polynomial interpolation (also called cubic interpolation), we choose the reliability given by

$$R(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Since we want to find the reliability at $t = 50$, and we are using a third order polynomial, we need to choose the four points closest to $t = 50$ that also bracket $t = 50$ to evaluate it.

The four points are $t_0 = 1$, $t_1 = 10$, $t_2 = 100$, and $t_3 = 1000$ hours.

$$t_0 = 1, \quad R(t_0) = 0.9999$$

$$t_1 = 10, \quad R(t_1) = 0.9998$$

$$t_2 = 100, \quad R(t_2) = 0.9980$$

$$t_3 = 1000, \quad R(t_3) = 0.9802$$

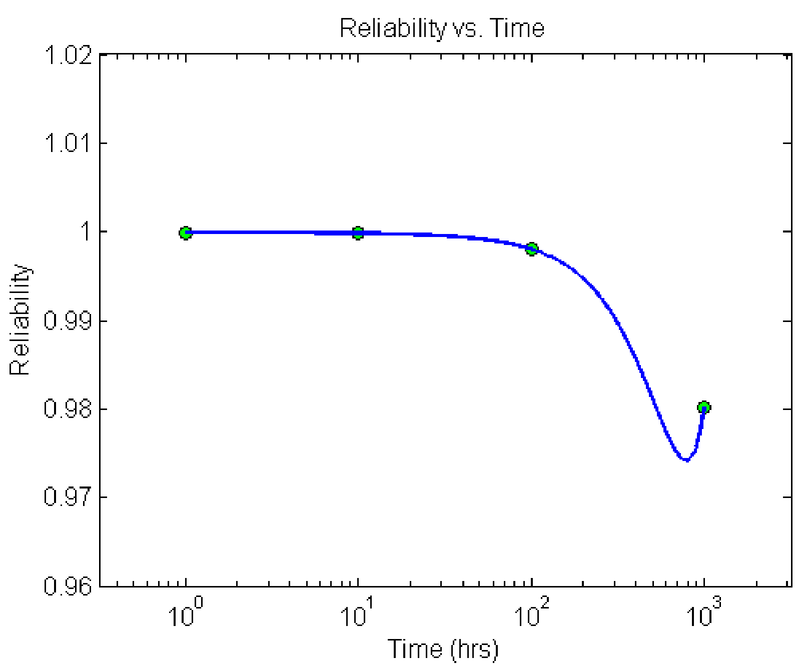

Figure 2 Graph of reliability as a function of time.

such that

$$R(1) = 0.9999 = a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3$$
$$R(10) = 0.9998 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3$$
$$R(100) = 0.9980 = a_0 + a_1(100) + a_2(100)^2 + a_3(100)^3$$
$$R(1000) = 0.9802 = a_0 + a_1(1000) + a_2(1000)^2 + a_3(1000)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 10 & 100 & 1000 \\ 1 & 100 & 10000 & 1\times 10^6 \\ 1 & 1000 & 1\times 10^6 & 1\times 10^9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.9999 \\ 0.9998 \\ 0.9980 \\ 0.9802 \end{bmatrix}$$

Solving the above gives

$$a_0 = 0.99991$$

$$a_1 = -1.0023\times 10^{-5}$$

$$a_2 = -9.9788\times 10^{-8}$$

$$a_3 = 9.0101\times 10^{-11}$$

Hence

$$R(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$= 0.99991 - 1.0023\times 10^{-5} t - 9.9788\times 10^{-8} t^2 + 9.0101\times 10^{-11} t^3, \quad 1 \le t \le 1000$$

The acceleration at $t = 50$ is given by

$$R'(50) = \frac{d}{dt} R(t)\Big|_{t=50}$$

Given that $R(t) = 0.99991 - 1.0023\times 10^{-5} t - 9.9788\times 10^{-8} t^2 + 9.0101\times 10^{-11} t^3, \quad 1 \le t \le 1000$,

$$R'(t) = \frac{d}{dt} R(t)$$
$$= \frac{d}{dt}\left(0.99991 - 1.0023\times 10^{-5} t - 9.9788\times 10^{-8} t^2 + 9.0101\times 10^{-11} t^3\right)$$
$$= -1.0023\times 10^{-5} - 1.9958\times 10^{-7} t + 2.7030\times 10^{-10} t^2, \quad 1 \le t \le 1000$$

$$R'(50) = -1.0023\times 10^{-5} - 1.9958\times 10^{-7}(50) + 2.7030\times 10^{-10}(50)^2$$

$$= -1.9326\times 10^{-5}$$

Using the same function, we can also calculate the value of $R(t)$ at $t = 50$.

$$R(t) = 0.99991 - 1.0023\times 10^{-5}t - 9.9788\times 10^{-8}t^2 + 9.0101\times 10^{-11}t^3, \quad 1 \le t \le 1000$$

$$R(50) = 0.99991 - 1.0023\times 10^{-5}(50) - 9.9788\times 10^{-8}(50)^2 + 9.0101\times 10^{-11}(50)^3$$

$$= 0.99917$$

The failure rate is then

$$h(t) = -\frac{R'(t)}{R(t)}$$

$$= -\frac{(-1.9326\times 10^{-5})}{0.99917}$$

$$= 1.9343\times 10^{-5}$$

Lagrange Polynomial

In this method, given $(x_0, y_0), \ldots, (x_n, y_n)$, one can fit a n^{th} order Lagrangian polynomial given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ and

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j = i$ omitted.

Then to find the first derivative, one can differentiate $f_n(x)$ once, and so on for other derivatives.

For example, the second order Lagrange polynomial passing through $(x_0, y_0), (x_1, y_1)$, and (x_2, y_2) is

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating the above equation gives

$$f_2'(x) = \frac{2x - (x_1 + x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2x - (x_0 + x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2x - (x_0 + x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating again would give the second derivative as

$$f_2''(x) = \frac{2}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Example 3

The failure rate $h(t)$ of a direct methanol fuel cell (DMFC) is given by the formula

$$h(t) = -\frac{R'(t)}{R(t)}$$

where $R(t)$ is the reliability at a certain time t, and the values of the reliability are given in Table 3.

Table 3 Reliability of DMFC system.

t (hrs)	0	1	10	100	1000	2000	3000	4000	5000
$R(t)$	1	0.9999	0.9998	0.9980	0.9802	0.9609	0.9419	0.9233	0.9050

Determine the value of the failure rate at $t = 50$ hours using second order Lagrangian polynomial interpolation for reliability.

Solution

For second order Lagrangian polynomial interpolation, we choose the reliability given by

$$R(t) = \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)R(t_0) + \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)R(t_1) + \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)R(t_2)$$

Since we want to find the reliability at $t = 50$, and we are using a second order Lagrangian polynomial, we need to choose the three points closest to $t = 50$ that also bracket $t = 50$ to evaluate it. The three points are $t_0 = 1$, $t_1 = 10$, and $t_2 = 100$.

Differentiating the above equation gives

$$R'(t) = \frac{2t - (t_1 + t_2)}{(t_0 - t_1)(t_0 - t_2)} R(t_0) + \frac{2t - (t_0 + t_2)}{(t_1 - t_0)(t_1 - t_2)} R(t_1) + \frac{2t - (t_0 + t_1)}{(t_2 - t_0)(t_2 - t_1)} R(t_2)$$

Hence

$$R'(50) = \frac{2(50) - (10 + 100)}{(1 - 10)(1 - 100)}(0.9999) + \frac{2(50) - (1 + 100)}{(10 - 1)(10 - 100)}(0.9998)$$
$$+ \frac{2(50) - (1 + 10)}{(100 - 1)(100 - 10)}(0.9980)$$
$$= -1.9102 \times 10^{-5}$$

We must also find the value of $R(t)$ at $t = 50$.

$$R(50) = \left(\frac{50 - 10}{1 - 10}\right)\left(\frac{50 - 100}{1 - 100}\right)(0.9999) + \left(\frac{50 - 1}{10 - 1}\right)\left(\frac{50 - 100}{10 - 100}\right)(0.9998)$$
$$+ \left(\frac{50 - 1}{100 - 1}\right)\left(\frac{50 - 10}{100 - 10}\right)(0.9980)$$
$$= 0.99918$$

The failure rate is then

$$h(t) = -\frac{R'(t)}{R(t)}$$

$$h(50) = -\frac{\left(-1.9102\times 10^{-5}\right)}{0.99918}$$

$$= 1.9118\times 10^{-5}$$

DIFFERENTIATION	
Topic	Discrete Functions-More Examples
Summary	Examples of Discrete Functions
Major	Industrial Engineering
Authors	Autar Kaw
Date	November 12, 2012
Web Site	http://numericalmethods.eng.usf.edu