Bisection Method

Industrial Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

Bisection Method

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Basis of Bisection Method

Theorem An equation f(x)=0, where f(x) is a real continuous function, has at least one root between x_l and x_u if $f(x_l)\,f(x_u) < 0$.

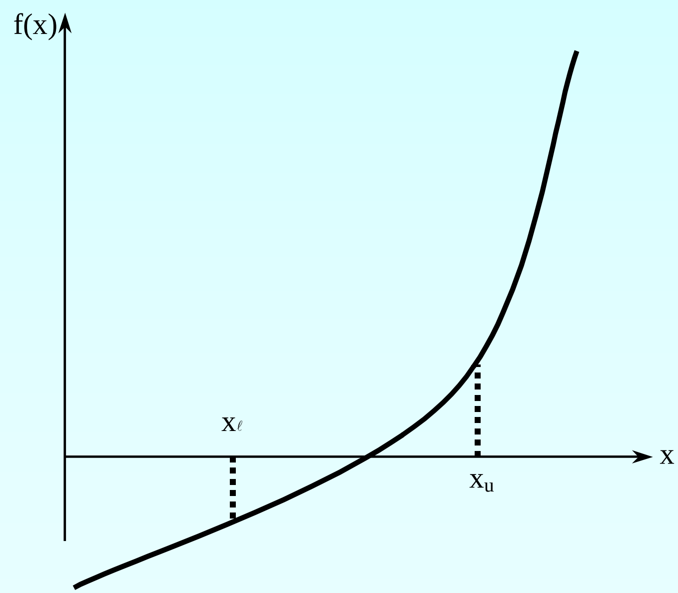


Figure 1 At least one root exists between the two points if the function is real, continuous, and changes sign.

Basis of Bisection Method

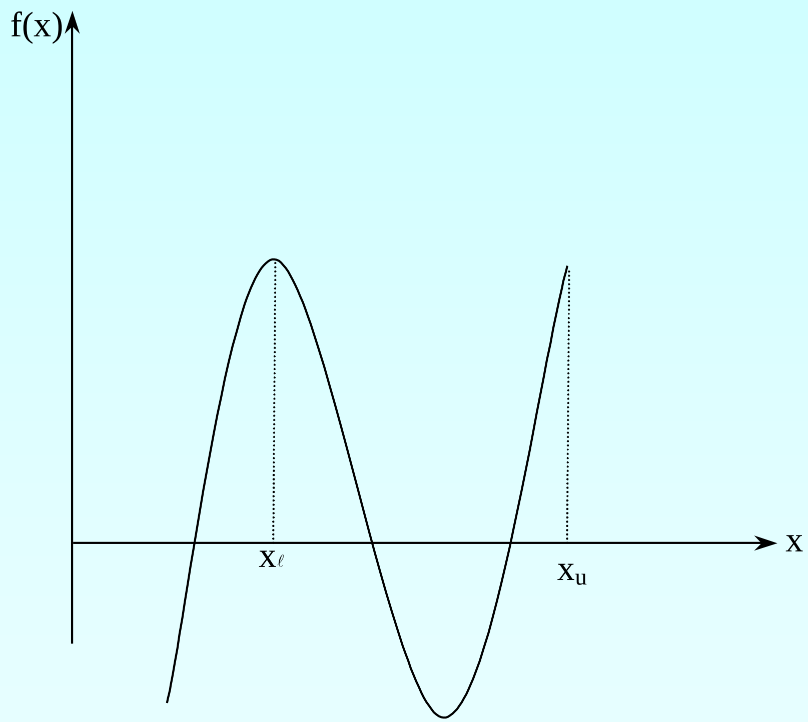


Figure 2 If function $f(x)$ does not change sign between two points, roots of the equation $f(x)=0$ may still exist between the two points.

Basis of Bisection Method

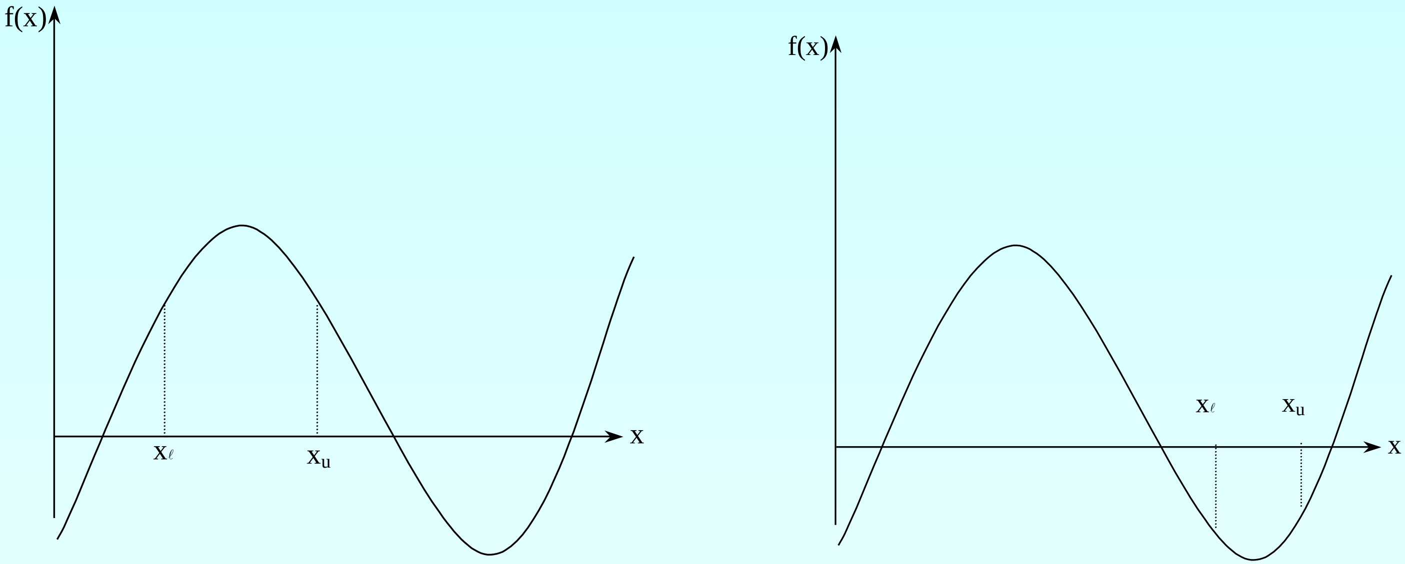


Figure 3 If the function $f(x)$ does not change sign between two points, there may not be any roots for the equation $f(x)=0$ between the two points.

Basis of Bisection Method

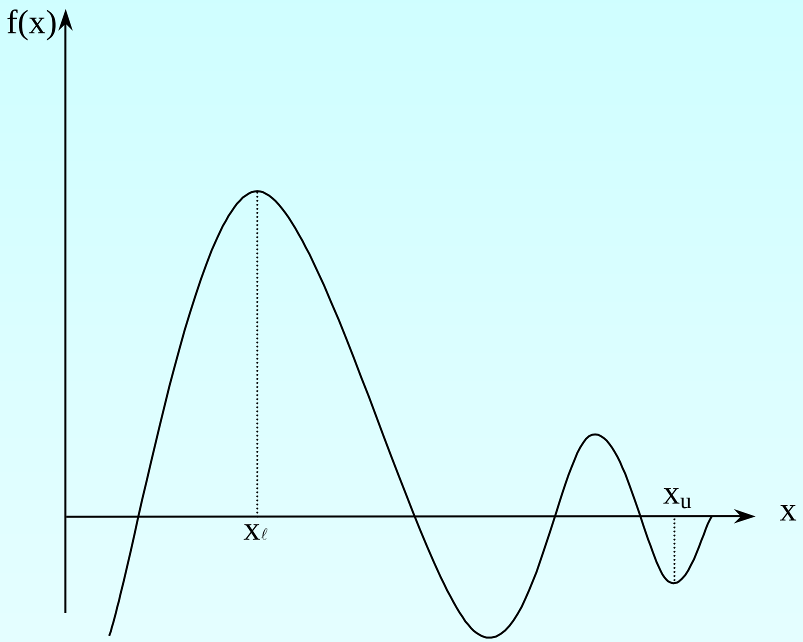


Figure 4 If the function $f(x)$ changes sign between two points, more than one root for the equation $f(x)=0$ may exist between the two points.

Algorithm for Bisection Method

Step 1

Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)\,f(x_u) < 0$, or in other words, f(x) changes sign between x_ℓ and x_u. This was demonstrated in Figure 1.

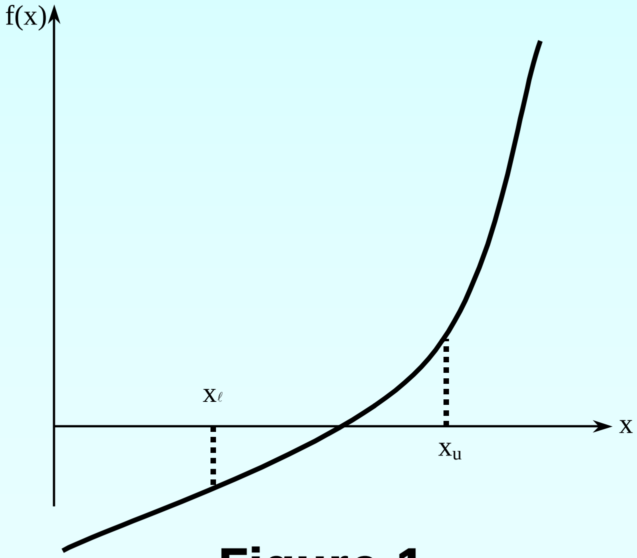


Figure 1

Step 2

Estimate the root, x_m of the equation f (x) = 0 as the mid point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$

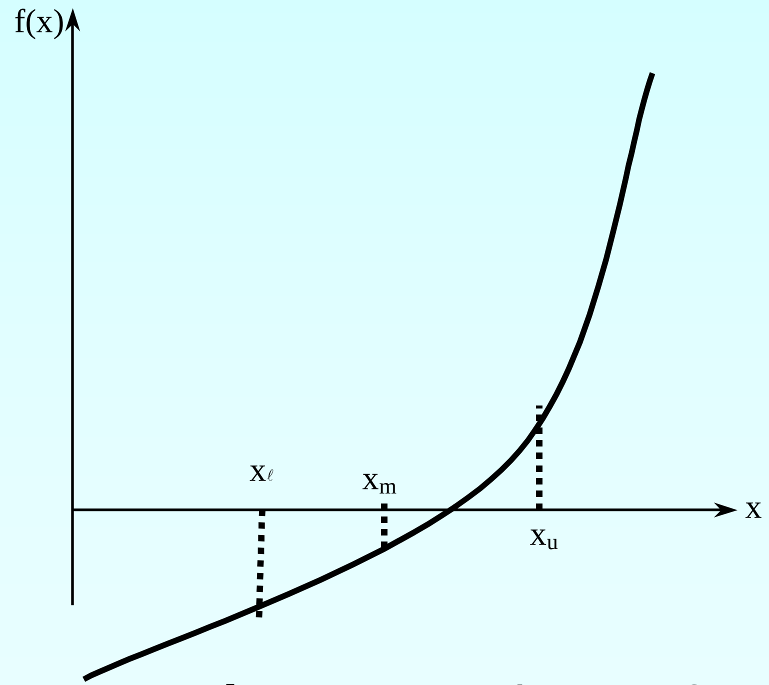


Figure 5 Estimate of x_m

Step 3

Now check the following

a) If $f(x_l)f(x_m) < 0$, then the root lies between x_ℓ and x_m; then $x_\ell = x_\ell$; $x_u = x_m$.

b) If $f(x_l)f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\ell = x_m$; $x_u = x_u$.

c) If $f(x_l)f(x_m) = 0$; then the root is x_m. Stop the algorithm if this is true.

Step 4

Find the new estimate of the root

$$x_m = \frac{x_\ell + x_u}{2}$$

Find the absolute relative approximate error

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

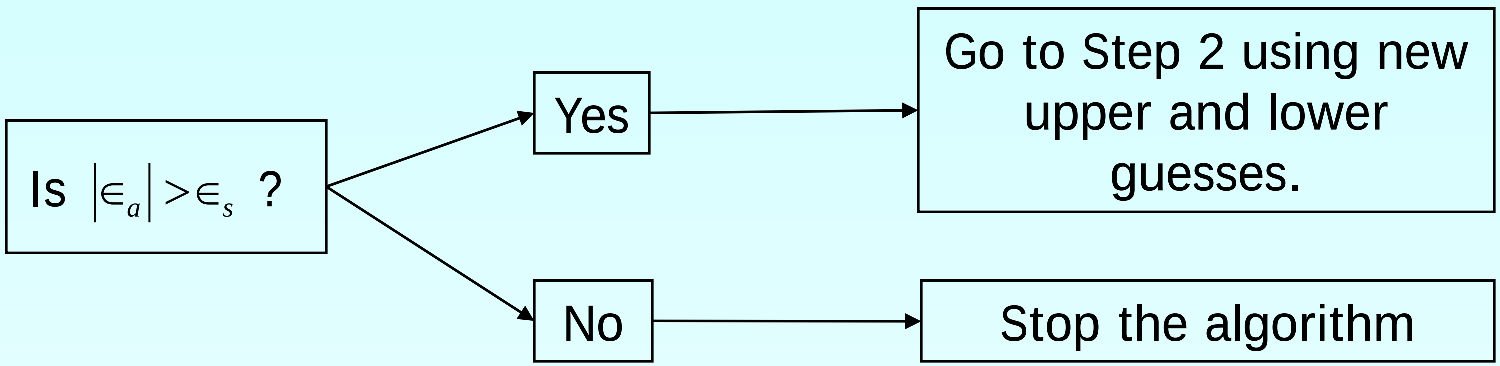
where

$$x_m^{old} = \text{previous estimate of root}$$

$$x_m^{new} = \text{current estimate of root}$$

Step 5

Compare the absolute relative approximate error $|\in_a|$ with the pre-specified error tolerance $\in_s$.



Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Example 1

You are working for a start-up computer assembly company and have been asked to determine the minimum number of computers that the shop will have to sell to make a profit.

The equation that gives the minimum number of computers ‘x’ to be sold after considering the total costs and the total sales is:

$$f(x) = 40x^{1.5} - 875x + 35000 = 0$$

Example 1 Cont.

Use the bisection method of finding roots of equations to find

- The minimum number of computers that need to be sold to make a profit. Conduct three iterations to estimate the root of the above equation.
- Find the absolute relative approximate error at the end of each iteration, and
- The number of significant digits at least correct at the end of each iteration.

Graph of function f(x)

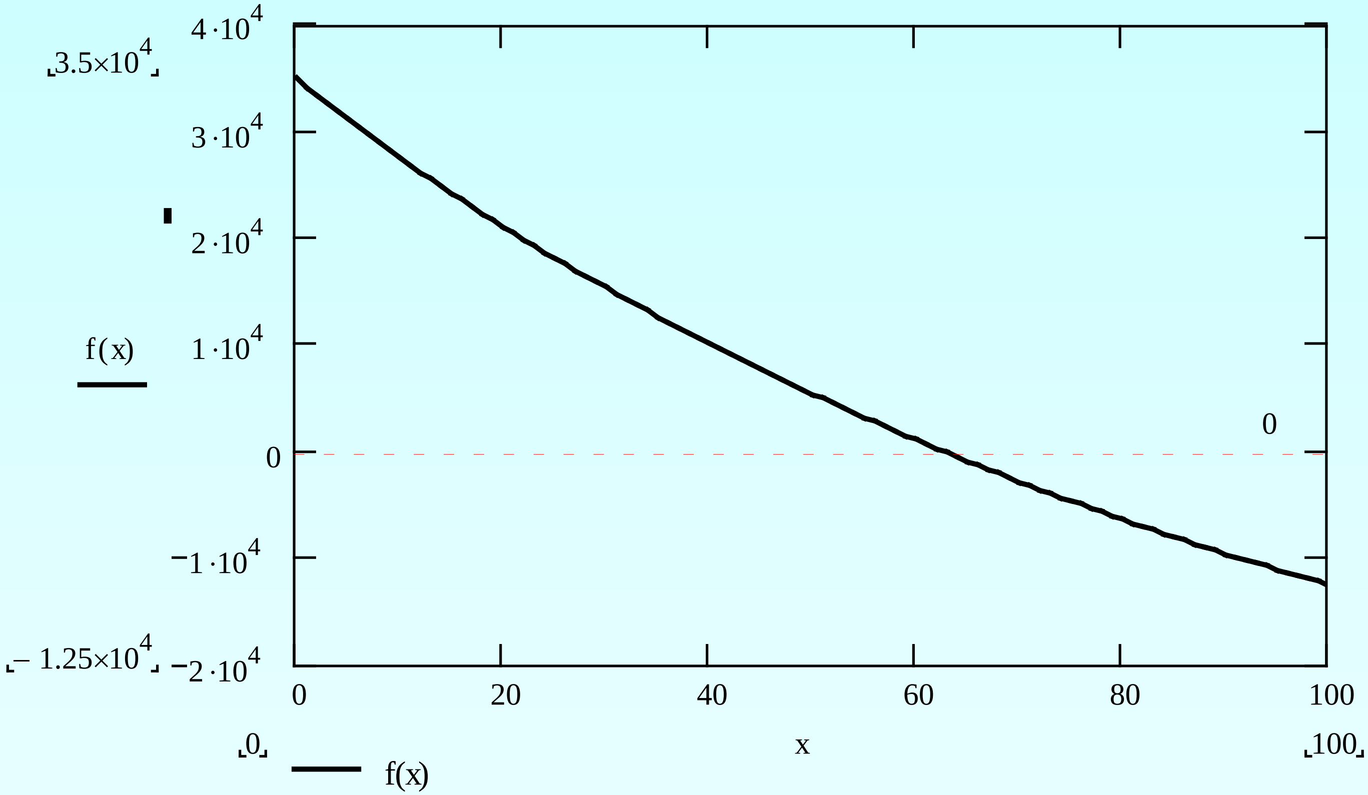


Figure 8 Graph of the function f(x).

$$f(x) = 40x^{1.5} - 875x + 35000 = 0$$

Example 1 Cont.

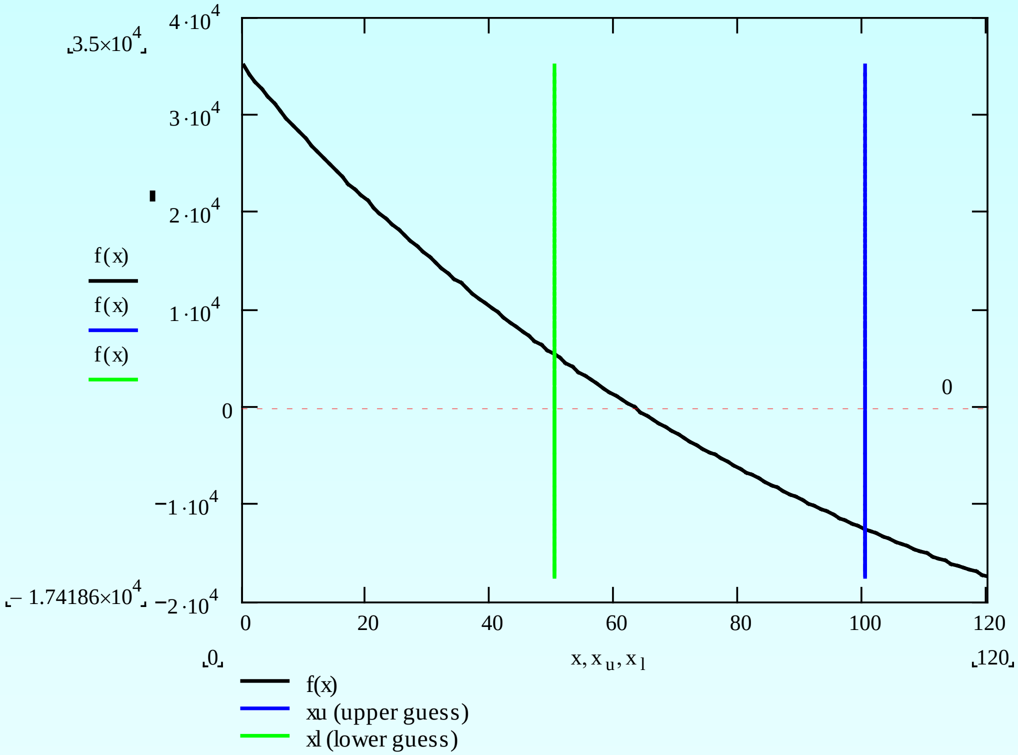


Figure 9 Checking the sign change between the limits.

Choose the bracket:

$$x_\ell = 50 \text{ and } x_u = 100$$

$$f(50) = 5392.1$$

$$f(100) = -12500$$

$$f(x_l)f(x_u) = f(50)f(100)$$
$$= (5392.1)(-12500) < 0$$

There is at least one root between x_ℓ and x_u.

Example 1 Cont.

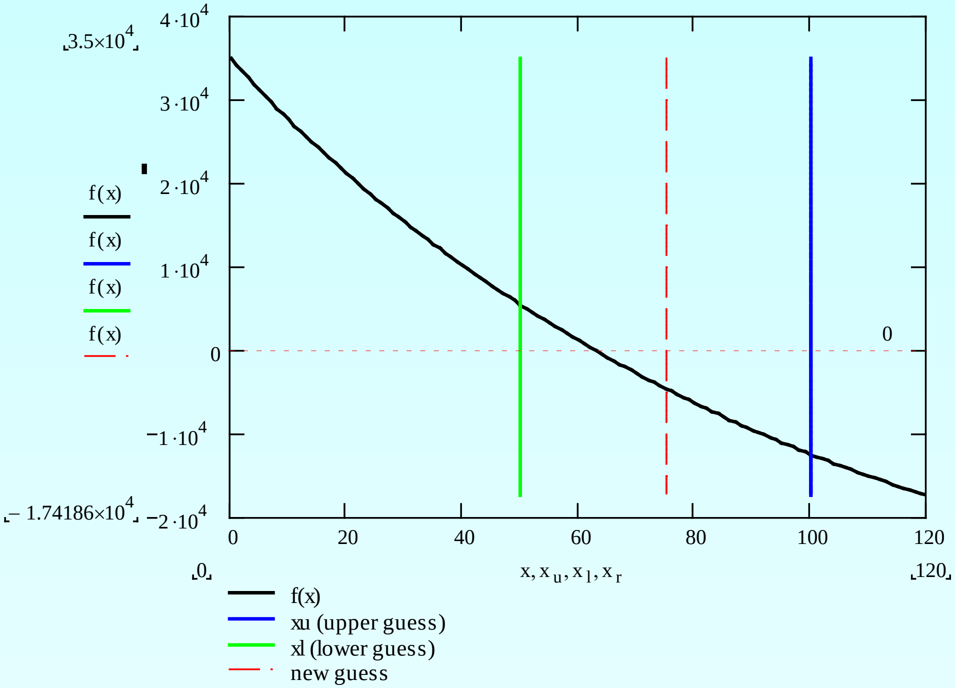


Figure 10 Graph of the estimate of the root after Iteration 1.

<u>Iteration 1</u>

The estimate of the root is

$$x_m = \frac{50+100}{2} = 75$$

$$f(75) = -4.6442\times10^3$$

$$f(x_l)f(x_m) = (5392.1)\left(-4.6442\times10^3\right) < 0$$

The root is bracketed between x_ℓ and x_m.

The new lower and upper limits of the new bracket are

$$x_\ell = 50 \text{ and } x_u = 75$$

At this point, the absolute relative approximate error cannot be calculated as we do not have a previous approximation.

Example 1 Cont.

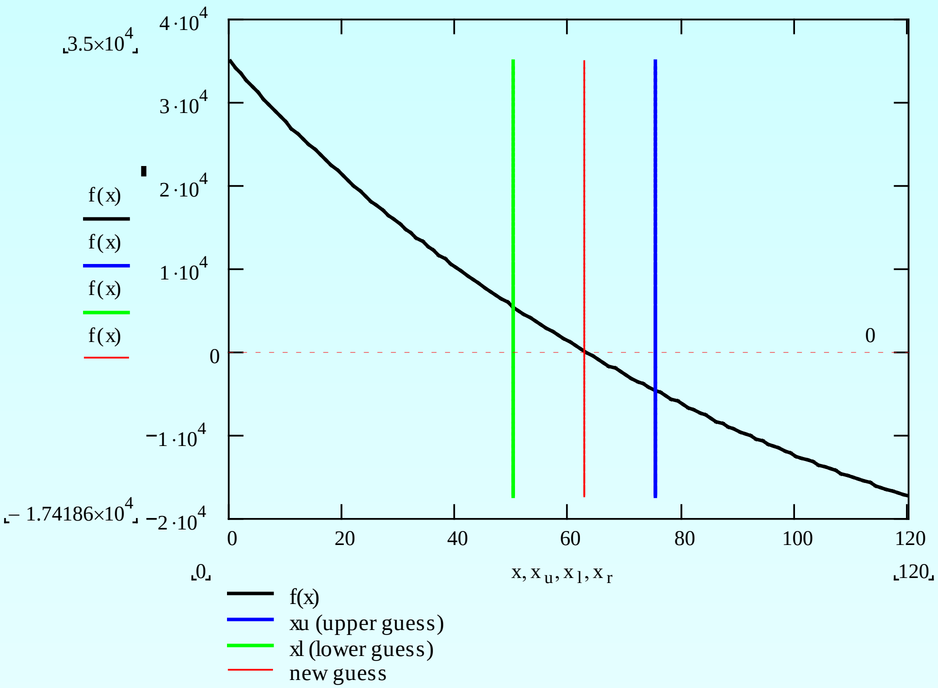


Figure 11 Graph of the estimate of the root after Iteration 2.

<u>Iteration 2</u>

The estimate of the root is

$$x_m = \frac{50+75}{2} = 62.5$$

$$f(62.5) = 76.735$$

$$f(x_l)f(x_m) = f(50)f(62.5) > 0$$

The root is bracketed between x_m and x_u .

The new lower and upper limits of the new bracket are

$$x_\ell = 62.5 \text{ and } x_u = 75$$

Example 1 Cont.

The absolute relative approximate error at the end of Iteration 2 is

$$|\in_a| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

$$= \left|\frac{62.5 - 75}{62.5}\right| \times 100$$

$$= 20\%$$

The number of significant digits that are at least correct in the estimated root is 0.

Example 1 Cont.

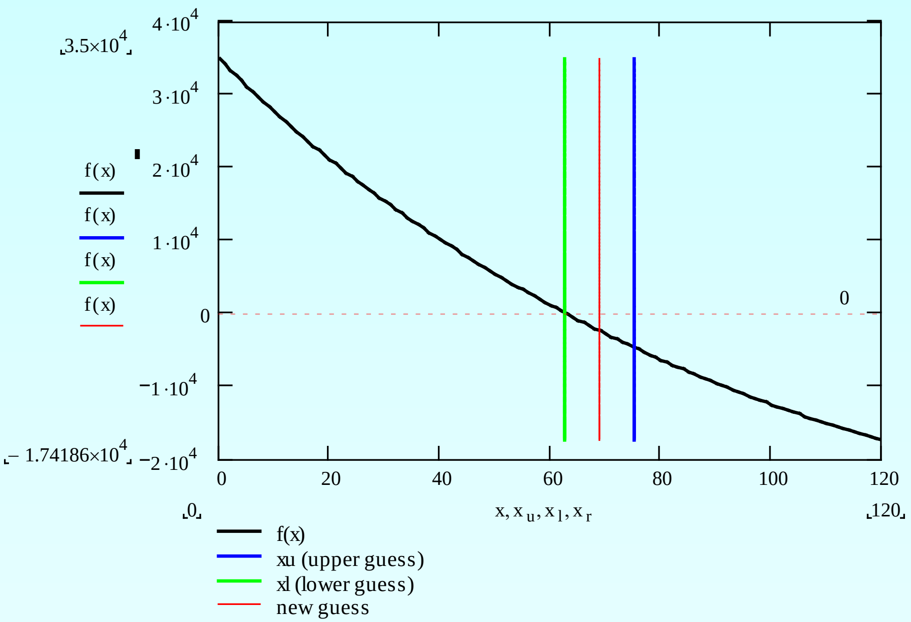


Figure 12 Graph of the estimate of the root after Iteration 3.

<u>Iteration 3</u>

The estimate of the root is

$$x_m = \frac{62.5 + 75}{2} = 68.75$$

$$f(68.75) = -2.3545 \times 10^3$$

$$f(x_l)f(x_m) = (76.735)(-2.3545 \times 10^3) < 0$$

The root is bracketed between x_l and x_m.

The new lower and upper limits of the new bracket are

$$x_\ell = 62.5 \text{ and } x_u = 68.75$$

Example 1 Cont.

The absolute relative approximate error at the end of Iteration 2 is

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

$$= \left|\frac{68.75 - 62.5}{68.75}\right| \times 100$$

$$= 9.0909\%$$

The number of significant digits that are at least correct in the estimated root is 0.

Convergence

Table 1 Root of f(x)=0 as function of number of iterations for bisection method.

Iteration	x_l	x_u	x_m	$\lvert\epsilon_a\rvert$ %	$f(x_m)$
1	50	100	75	----------	$-4.6442\ \ 10^3$
2	50	75	62.5	20	76.735
3	62.5	75	68.75	9.0909	$-2.3545\ \ 10^3$
4	62.5	68.75	65.625	4.7619	$-1.1569\ \ 10^3$
5	62.5	65.625	64.063	2.4390	−544.68
6	62.5	64.063	63.281	1.2346	−235.12
7	62.5	63.281	62.891	0.62112	−79.483
8	62.5	62.891	62.695	0.31153	−1.4459
9	62.5	62.695	62.598	0.15601	37.627
10	62.598	62.695	62.646	0.077942	18.086

Advantages

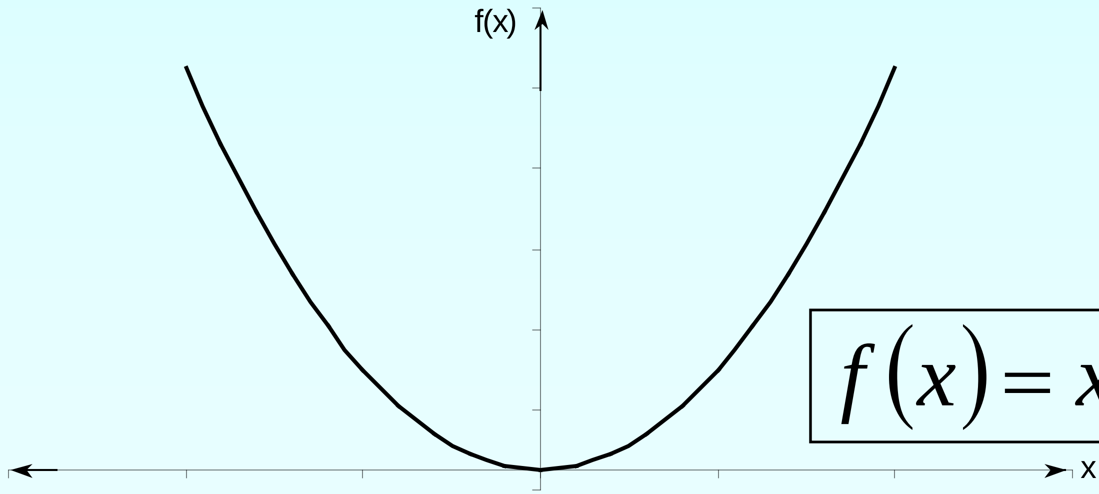
- Always convergent
- The root bracket gets halved with each iteration - guaranteed.

Drawbacks

- Slow convergence
- If one of the initial guesses is close to the root, the convergence is slower

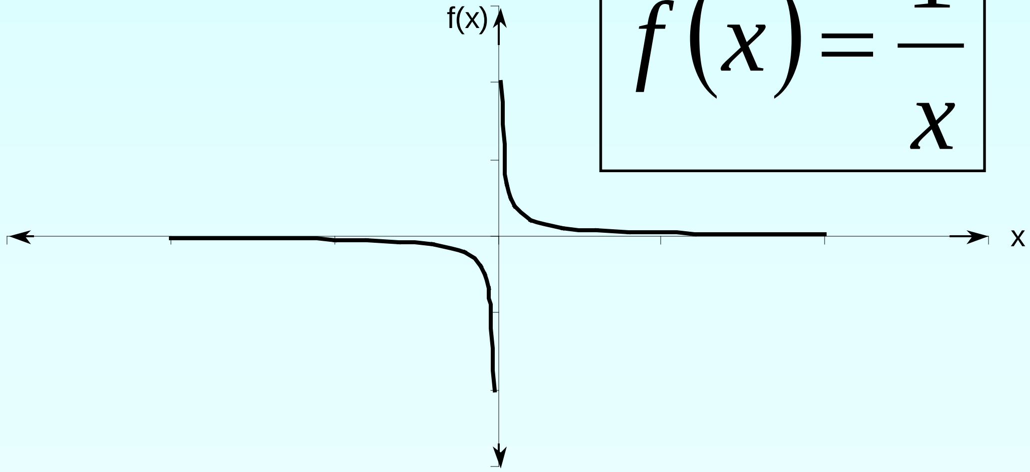
Drawbacks (continued)

- If a function f(x) is such that it just touches the x-axis it will be unable to find the lower and upper guesses.



Drawbacks (continued)

- Function changes sign but root does not exist



$$f(x)=\frac{1}{x}$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/bisection_method.html

THE END

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