Lagrangian Interpolation

Industrial Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

Lagrange Method of Interpolation

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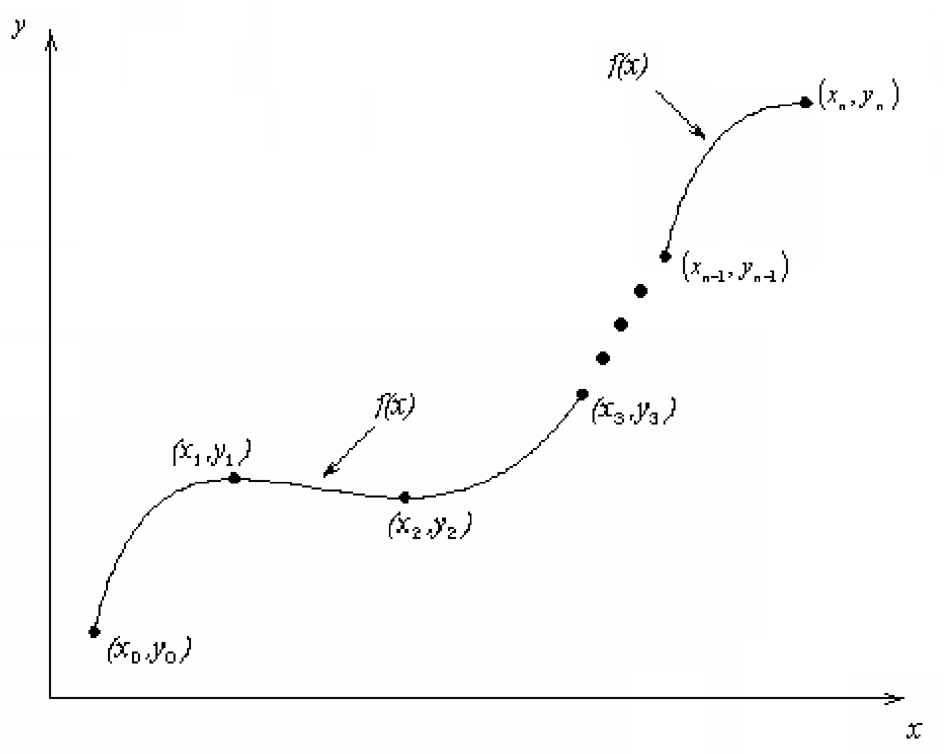
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

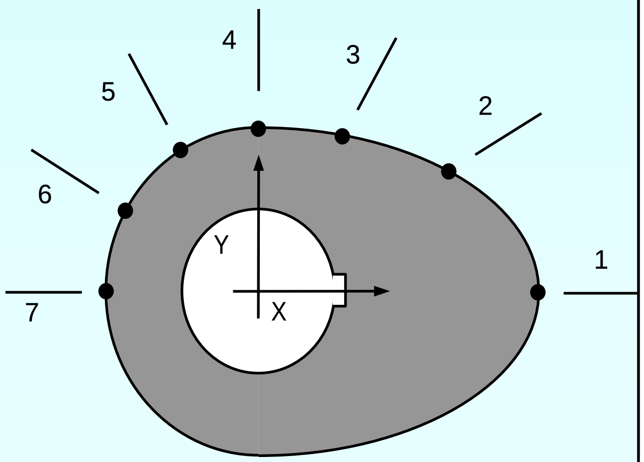
$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

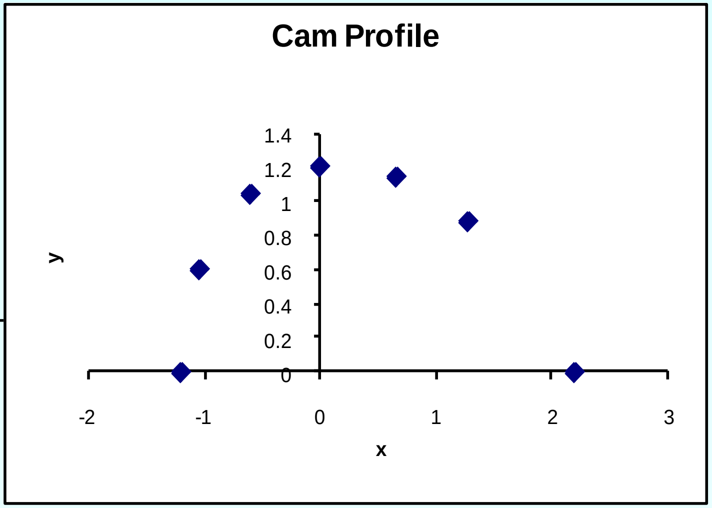
$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a straight line profile between x = 1.28 to x = 0.66, what is the value of y at x=1.1 ? Find using the Lagrange method and linear interpolation.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00





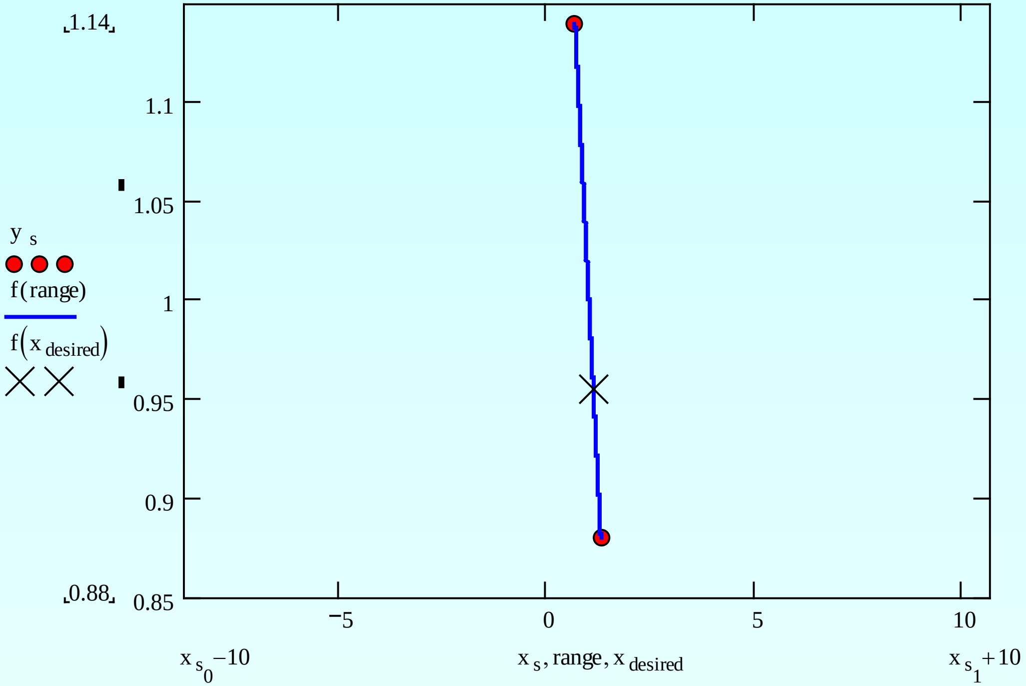
Linear Interpolation

$$y(x) = \sum_{i=0}^{1} L_i(x) y(x_i)$$

$$= L_0(x) y(x_0) + L_1(x) y(x_1)$$

$$x_0 = 1.28, y(x_0) = 0.88$$

$$x_1 = 0.66, y(x_1) = 1.14$$



Linear Interpolation (contd)

$$L_0(x)=\prod_{\substack{j=0\\ j\neq 0}}^{1}\frac{x-x_j}{x_0-x_j}=\frac{x-x_1}{x_0-x_1}$$

$$L_0(x)=\prod_{\substack{j=0\\ j\neq 1}}^{1}\frac{x-x_j}{x_0-x_j}=\frac{x-x_0}{x_1-x_0}$$

$$y(x)=\frac{x-x_1}{x_0-x_1}y(x_0)+\frac{x-x_0}{x_1-x_0}y(x_1)$$

$$=\frac{x-0.66}{1.28-0.66}(0.88)+\frac{x-1.28}{0.66-1.28}(1.14),\quad 0.66\le x\le 1.28$$

$$y(1.10)=\frac{1.10-0.66}{1.28-0.66}(0.88)+\frac{1.10-1.28}{0.66-1.28}(1.14)$$

$$=0.70968(0.88)+0.29032(1.14)$$

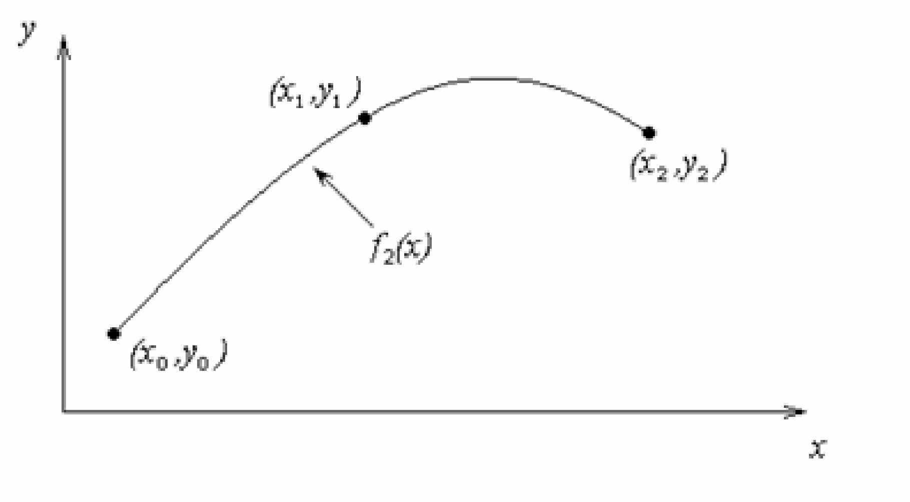
$$=0.95548\text{ in.}$$

Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the velocity given by

$$v(t) = \sum_{i=0}^{2} L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2)$$



Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a straight line profile between x = 1.28 to x = 0.66, what is the value of y at x=1.1 ? Find using the Lagrange method and quadratic interpolation.

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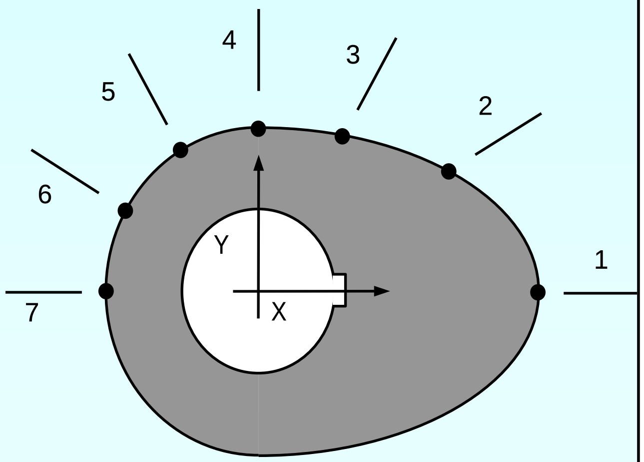
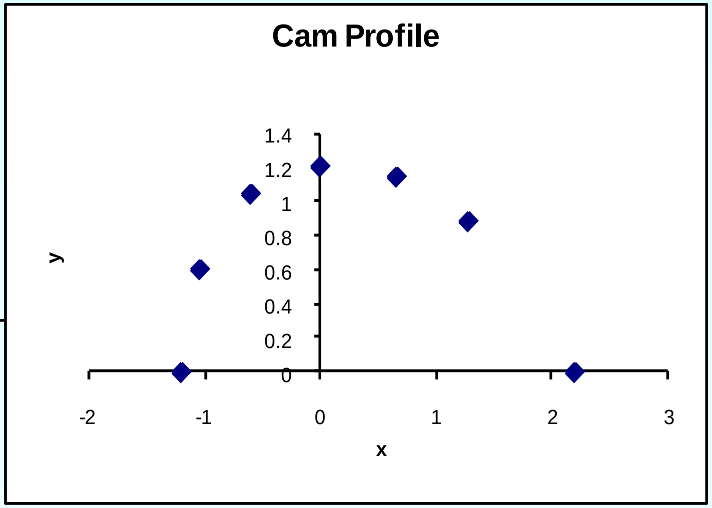
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00





Quadratic Interpolation (contd)

$$x_o = 2.20, \quad y(x_o) = 0.00$$

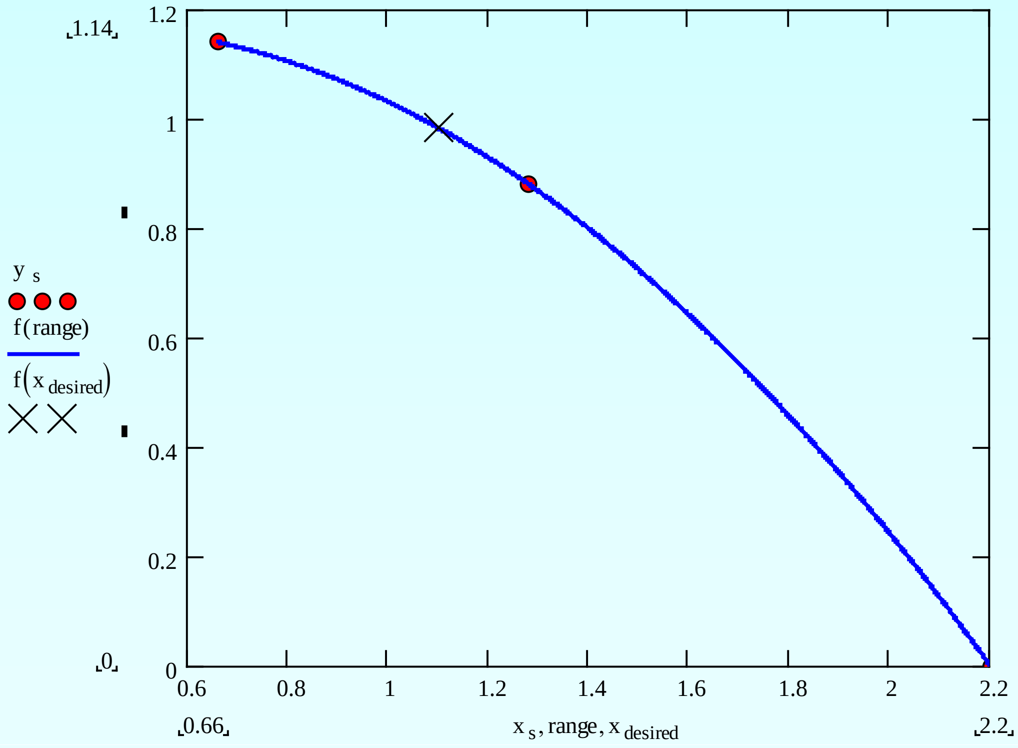
$$x_1 = 1.28, \quad y(x_1) = 0.88$$

$$x_2 = 0.66, \quad y(x_2) = 1.14$$

$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^{2} \frac{x - x_j}{x_0 - x_j} = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^{2} \frac{x - x_j}{x_1 - x_j} = \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)$$

$$L_2(x) = \prod_{\substack{j=0 \\ j \neq 2}}^{2} \frac{x - x_j}{x_2 - x_j} = \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)$$



Quadratic Interpolation (contd)

$$y(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)y(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)y(x_1) + \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)y(x_2)$$

$$x_0 \le x \le x_2$$

$$y(1.10) = \left(\frac{1.10 - 1.28}{2.20 - 1.28}\right)\left(\frac{1.10 - 0.66}{2.20 - 0.66}\right)(0.00) + \left(\frac{1.10 - 2.20}{1.28 - 2.20}\right)\left(\frac{1.10 - 0.66}{1.28 - 0.66}\right)(0.88)$$

$$+ \left(\frac{1.10 - 2.20}{0.66 - 2.20}\right)\left(\frac{1.10 - 1.28}{0.66 - 1.28}\right)(1.14)$$

$$= (-0.055901)(0.00) + (0.84853)(0.88) + (0.20737)(1.14)$$

$$= 0.98311\text{in.}$$

The absolute relative approximate error obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{0.98311 - 0.95548}{0.98311}\right| \times 100$$

$$= 2.8100\%$$

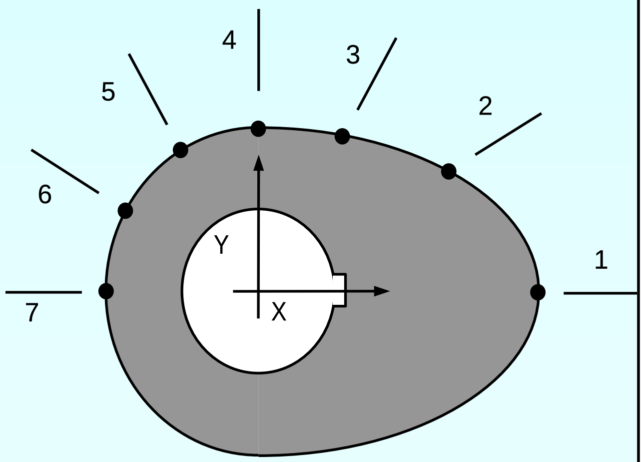
Comparison Table

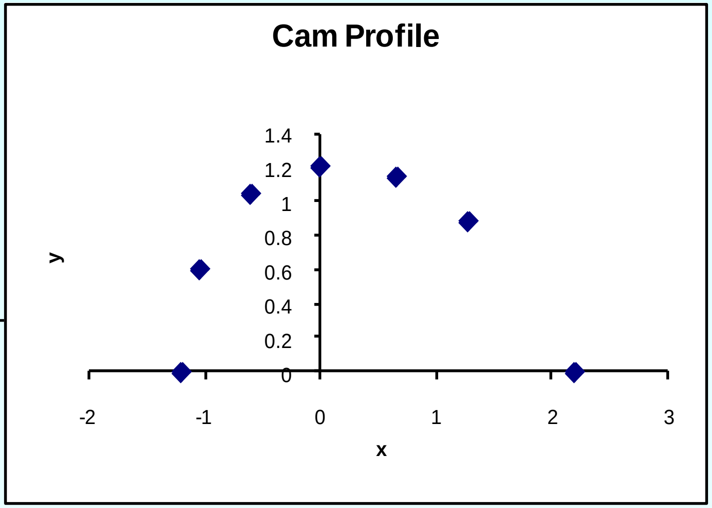
Order of Polynomial	**1**	**2**
Value of y at x=1.1	0.95548	0.98311
Absolute Relative Approximate Error	----------	2.8100 %

Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a straight line profile between x = 1.28 to x = 0.66, what is the value of y at x=1.1 ? Find using the Lagrange method and sixth order polynomial.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00





Sixth Order Interpolation

$$y(x) = \sum_{i=0}^{6} L_i(x) y(x_i)$$

$$= L_0(x)y(x_0) + L_1(x)y(x_1) + L_2(x)y(x_2) + L_3(x)y(x_3) + L_4(x)y(x_4) + L_5(x)y(x_5) + L_6(x)y(x_6)$$

$$x_o = 2.20, \quad y(x_o) = 0.00$$

$$x_1 = 1.28, \quad y(x_1) = 0.88$$

$$x_2 = 0.66, \quad y(x_2) = 1.14$$

$$x_3 = 0.00, \quad y(x_3) = 1.20$$

$$x_4 = -0.60, \quad y(x_4) = 1.04$$

$$x_5 = -1.04, \quad y(x_5) = 0.60$$

$$x_6 = -1.20, \quad y(x_6) = 0.00$$

Sixth Order Interpolation (contd)

$$L_0(x)=\prod_{\substack{j=0\\j\neq 0}}^{6}\frac{x-x_j}{x_0-x_j}=\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)\left(\frac{x-x_3}{x_0-x_3}\right)\left(\frac{x-x_4}{x_0-x_4}\right)\left(\frac{x-x_5}{x_0-x_5}\right)\left(\frac{x-x_6}{x_0-x_6}\right)$$

$$L_1(x)=\prod_{\substack{j=0\\j\neq 1}}^{6}\frac{x-x_j}{x_1-x_j}=\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)\left(\frac{x-x_4}{x_1-x_4}\right)\left(\frac{x-x_5}{x_1-x_5}\right)\left(\frac{x-x_6}{x_1-x_6}\right)$$

$$L_2(x)=\prod_{\substack{j=0\\j\neq 2}}^{6}\frac{x-x_j}{x_2-x_j}=\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right)\left(\frac{x-x_4}{x_2-x_4}\right)\left(\frac{x-x_5}{x_2-x_5}\right)\left(\frac{x-x_6}{x_2-x_6}\right)$$

$$L_3(x)=\prod_{\substack{j=0\\j\neq 3}}^{6}\frac{x-x_j}{x_3-x_j}=\left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)\left(\frac{x-x_4}{x_3-x_4}\right)\left(\frac{x-x_5}{x_3-x_5}\right)\left(\frac{x-x_6}{x_3-x_6}\right)$$

$$L_4(x)=\prod_{\substack{j=0\\j\neq 4}}^{6}\frac{x-x_j}{x_4-x_j}=\left(\frac{x-x_0}{x_4-x_0}\right)\left(\frac{x-x_1}{x_4-x_1}\right)\left(\frac{x-x_2}{x_4-x_2}\right)\left(\frac{x-x_3}{x_4-x_3}\right)\left(\frac{x-x_5}{x_4-x_5}\right)\left(\frac{x-x_6}{x_4-x_6}\right)$$

$$L_5(x)=\prod_{\substack{j=0\\j\neq 5}}^{6}\frac{x-x_j}{x_5-x_j}=\left(\frac{x-x_0}{x_5-x_0}\right)\left(\frac{x-x_1}{x_5-x_1}\right)\left(\frac{x-x_2}{x_5-x_2}\right)\left(\frac{x-x_3}{x_5-x_3}\right)\left(\frac{x-x_4}{x_5-x_4}\right)\left(\frac{x-x_6}{x_5-x_6}\right)$$

$$L_6(x)=\prod_{\substack{j=0\\j\neq 6}}^{6}\frac{x-x_j}{x_6-x_j}=\left(\frac{x-x_0}{x_6-x_0}\right)\left(\frac{x-x_1}{x_6-x_1}\right)\left(\frac{x-x_2}{x_6-x_2}\right)\left(\frac{x-x_3}{x_6-x_3}\right)\left(\frac{x-x_4}{x_6-x_4}\right)\left(\frac{x-x_5}{x_6-x_5}\right)$$

Sixth Order Polynomial (contd)

$$y(x)=\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)\left(\frac{x-x_3}{x_0-x_3}\right)\left(\frac{x-x_4}{x_0-x_4}\right)\left(\frac{x-x_5}{x_0-x_5}\right)\left(\frac{x-x_6}{x_0-x_6}\right)y(x_0)$$

$$+\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)\left(\frac{x-x_4}{x_1-x_4}\right)\left(\frac{x-x_5}{x_1-x_5}\right)\left(\frac{x-x_6}{x_1-x_6}\right)y(x_1)$$

$$+\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right)\left(\frac{x-x_4}{x_2-x_4}\right)\left(\frac{x-x_5}{x_2-x_5}\right)\left(\frac{x-x_6}{x_2-x_6}\right)y(x_2)$$

$$+\left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)\left(\frac{x-x_4}{x_3-x_4}\right)\left(\frac{x-x_5}{x_3-x_5}\right)\left(\frac{x-x_6}{x_3-x_6}\right)y(x_3)$$

$$+\left(\frac{x-x_0}{x_4-x_0}\right)\left(\frac{x-x_1}{x_4-x_1}\right)\left(\frac{x-x_2}{x_4-x_2}\right)\left(\frac{x-x_3}{x_4-x_3}\right)\left(\frac{x-x_5}{x_4-x_5}\right)\left(\frac{x-x_6}{x_4-x_6}\right)y(x_4)$$

$$+\left(\frac{x-x_0}{x_5-x_0}\right)\left(\frac{x-x_1}{x_5-x_1}\right)\left(\frac{x-x_2}{x_5-x_2}\right)\left(\frac{x-x_3}{x_5-x_3}\right)\left(\frac{x-x_4}{x_5-x_4}\right)\left(\frac{x-x_6}{x_5-x_6}\right)y(x_5)$$

$$+\left(\frac{x-x_0}{x_6-x_0}\right)\left(\frac{x-x_1}{x_6-x_1}\right)\left(\frac{x-x_2}{x_6-x_2}\right)\left(\frac{x-x_3}{x_6-x_3}\right)\left(\frac{x-x_4}{x_6-x_4}\right)\left(\frac{x-x_5}{x_6-x_5}\right)y(x_6)$$

Sixth Order Polynomial (contd)

$$y(x)=\frac{(x-1.28)(x-0.66)(x-0.00)(x+0.60)(x+1.04)(x+1.20)}{(2.20-1.28)(2.20-0.66)(2.20-0.00)(2.20+0.60)(2.20+1.04)(2.20+1.20)}(0.00)$$

$$+\frac{(x-2.20)(x-0.66)(x-0.00)(x+0.60)(x+1.04)(x+1.20)}{(1.28-2.20)(1.28-0.66)(1.28-0.00)(1.28+0.60)(1.28+1.04)(1.28+1.20)}(0.88)$$

$$+\frac{(x-2.20)(x-1.28)(x-0.00)(x+0.60)(x+1.04)(x+1.20)}{(0.66-2.20)(0.66-1.28)(0.66-0.00)(0.66+0.60)(0.66+1.04)(0.66+1.20)}(1.14)$$

$$+\frac{(x-2.20)(x-1.28)(x-0.66)(x+0.60)(x+1.04)(x+1.20)}{(0.00-2.20)(0.00-1.28)(0.00-0.66)(0.00+0.60)(0.00+1.04)(0.00+1.20)}(1.20)$$

$$+\frac{(x-2.20)(x-1.28)(x-0.66)(x-0.00)(x+1.04)(x+1.20)}{(-0.60-2.20)(-0.60-1.28)(-0.60-0.66)(-0.60-0.00)(-0.60+1.04)(-0.60+1.20)}(1.04)$$

$$+\frac{(x-2.20)(x-1.28)(x-0.66)(x-0.00)(x+0.60)(x+1.20)}{(-1.04-2.20)(-1.04-1.28)(-1.04-0.66)(-1.04-0.00)(-1.04+0.60)(-1.04+1.20)}(0.60)$$

$$+\frac{(x-2.20)(x-1.28)(x-0.66)(x-0.00)(x+0.60)(x+1.04)}{(-1.20-2.20)(-1.20-1.28)(-1.20-0.66)(-1.20-0.00)(-1.20+0.60)(-1.20+1.04)}(0.00)$$

Sixth Order Polynomial (contd)

$$= \frac{x^6 - 0.02x^5 - 4.0784x^4 - 2.5406x^3 + 1.6220x^2 + 1.0873x}{-8.9744}$$

$$+ \frac{x^6 - 0.64x^5 - 4.4752x^4 - 0.27392x^3 + 4.6932x^2 + 2.1086x}{2.2023}$$

$$+ \frac{x^6 - 1.3x^5 - 4.0528x^4 + 2.6797x^3 + 4.8740x^2 - 0.98892x - 1.3917}{-1.15974}$$

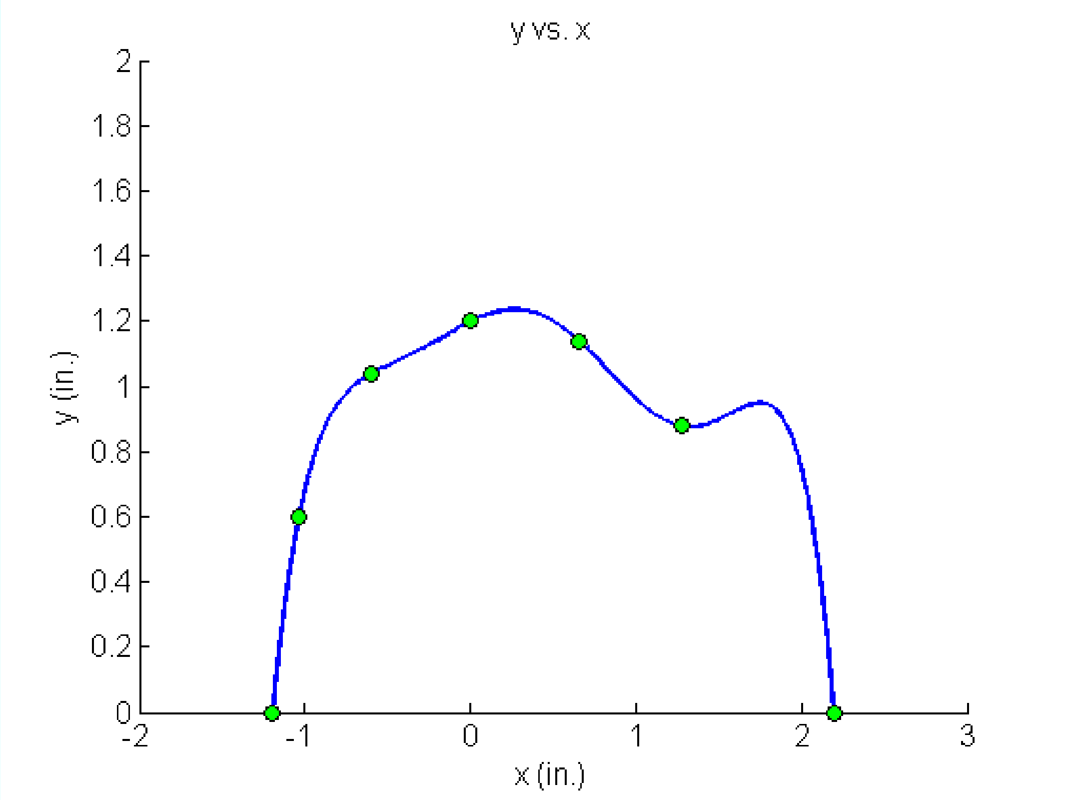
$$+ \frac{x^6 - 1.9x^5 - 2.9128x^4 + 4.4274x^3 + 2.2176x^2 - 2.31948x}{1.0102}$$

$$+ \frac{x^6 - 2.34x^5 - 1.6192x^4 + 4.3637x^3 + 0.33581x^2 - 1.3382x}{-1.5593}$$

$$y(x) = 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3$$

$$+ 0.072013x^4 + 0.45241x^5 + 0.17103x^6, \; -1.20 \le x \le 2.20$$

Sixth Order Polynomial (contd)



$$y(x) = 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3 + 0.072013x^4 + 0.45241x^5 + 0.17103x^6, \quad -1.20 \le x \le 2.20$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/lagrange_method.html

THE END

http://numericalmethods.eng.usf.edu