Newton's Divided Difference Polynomial Method of Interpolation

Industrial Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

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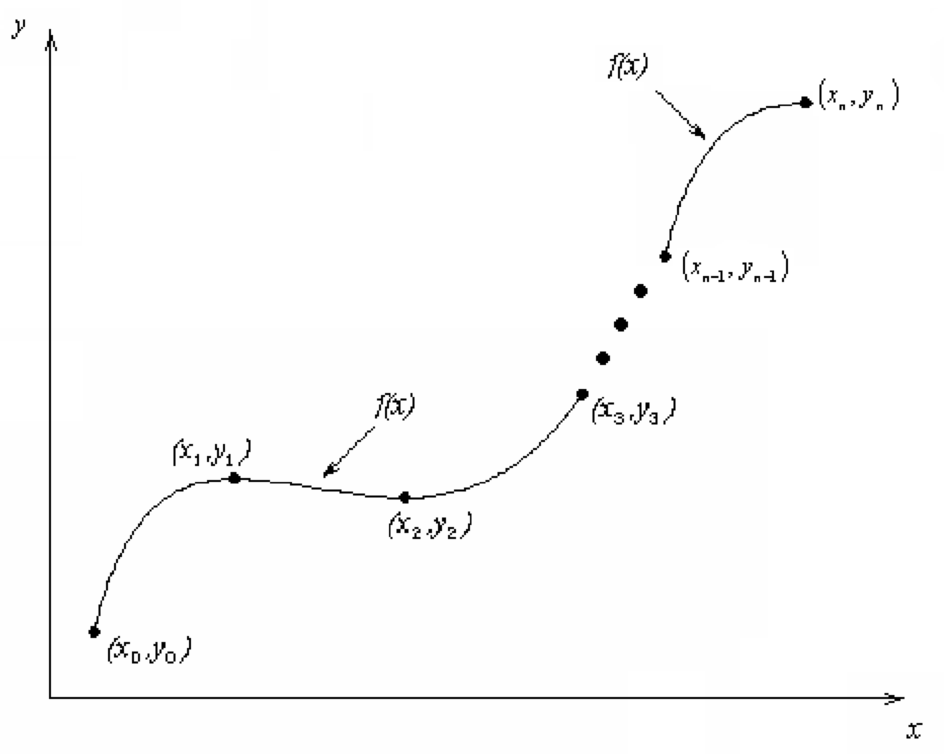
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

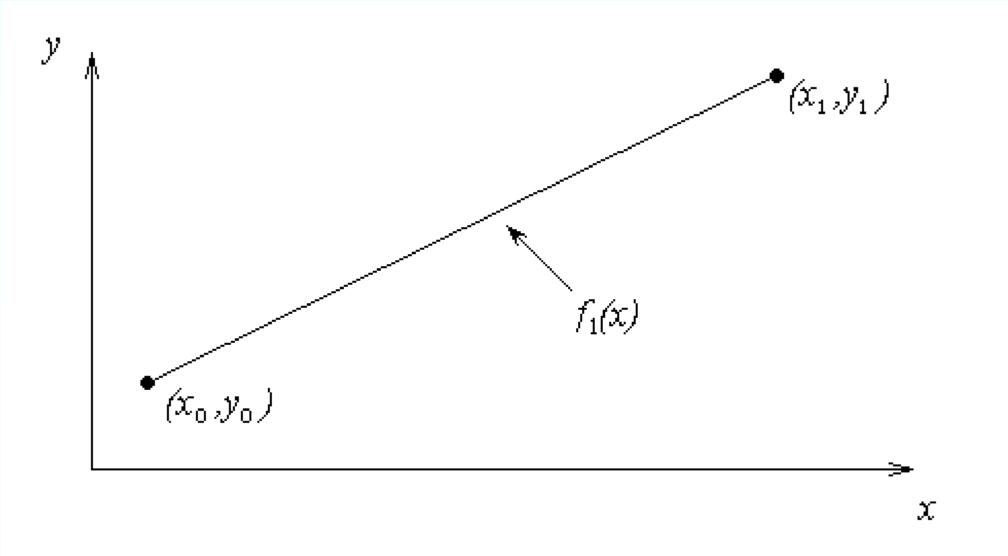
Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Newton's Divided Difference Method

<u>Linear interpolation</u>: Given $(x_0, y_0), (x_1, y_1),$ pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$



where

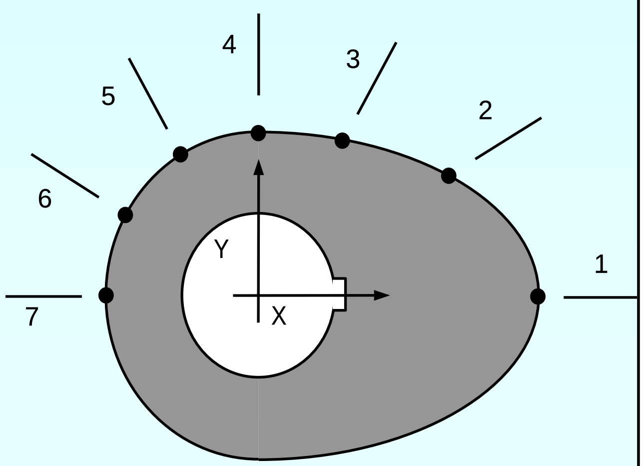
$$b_0 = f(x_0)$$

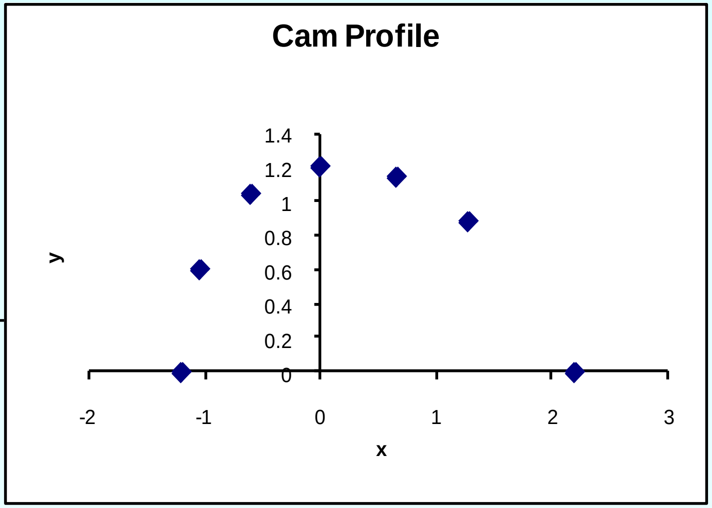
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a straight line profile between x = 1.28 to x = 0.66, what is the value of y at x=1.1 ? Find using the Newton's divided difference method for linear interpolation.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00





Linear Interpolation

$$y(x) = b_0 + b_1(x - x_0)$$

$$x_0 = 1.28,\ y(x_0) = 0.88$$

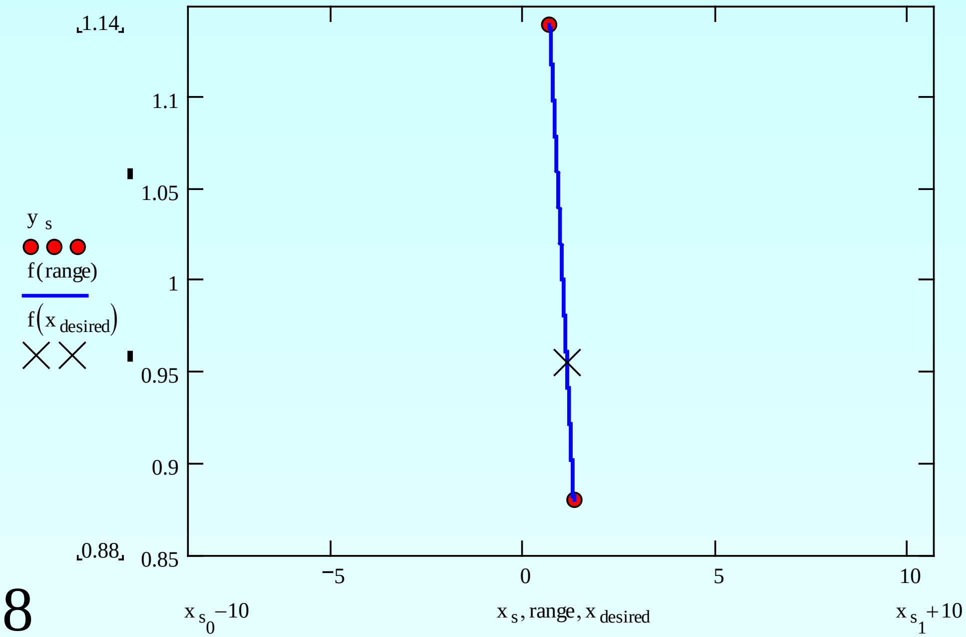
$$x_1 = 0.66,\ y(x_1) = 1.14$$

$$b_0 = y(x_0)$$

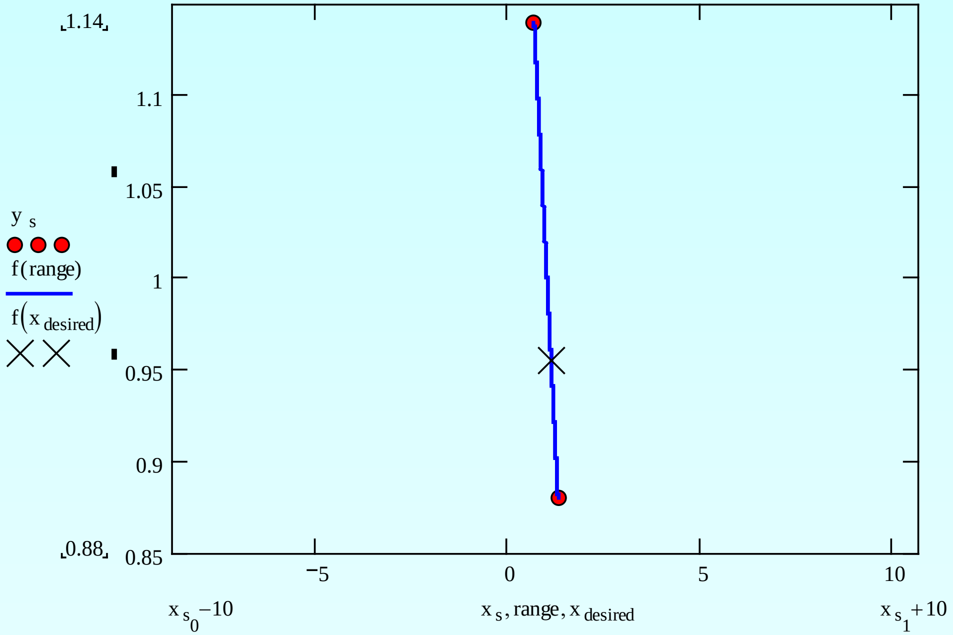
$$= 0.88$$

$$b_1 = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{1.14 - 0.88}{0.66 - 1.28}$$

$$= -0.41935$$



Linear Interpolation (contd)



$$y(x) = b_0 + b_1(x - x_0)$$

$$= 0.88 - 0.41935(x - 1.28), \qquad 0.66 \le x \le 1.28$$

At $x = 1.10$

$$y(1.10) = 0.88 - 0.41935(1.10 - 1.28)$$

$$= 0.95548 \; in.$$

Quadratic Interpolation

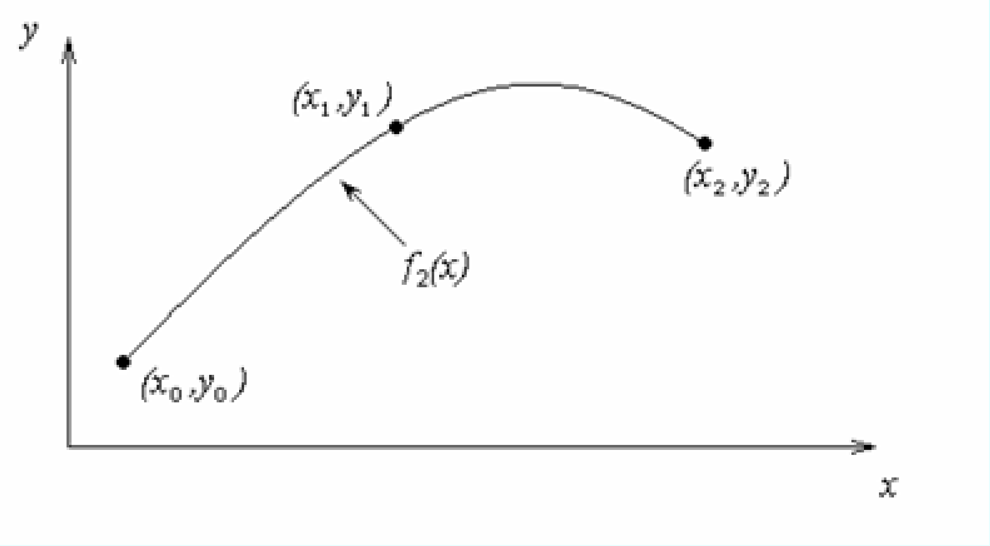
Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

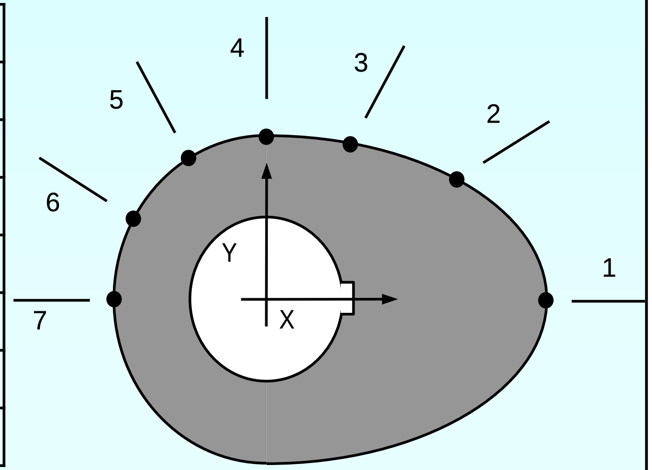
$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

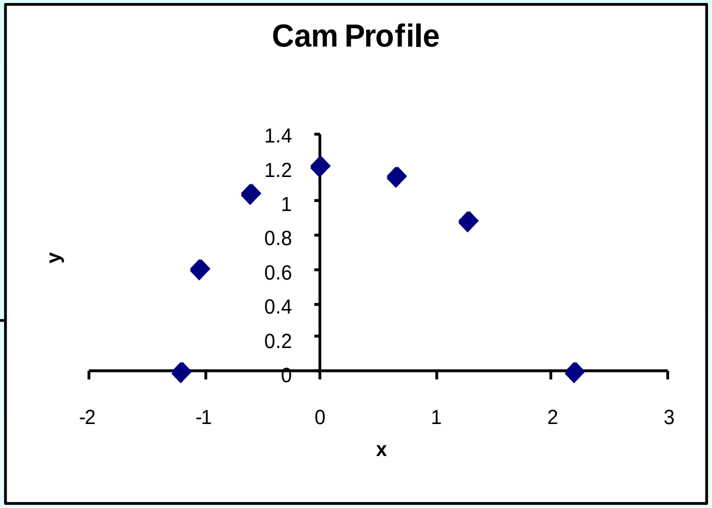


Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a straight line profile between x = 1.28 to x = 0.66, what is the value of y at x=1.1 ? Find using the Newton's divided difference method for quadratic interpolation.

Point	*x* (in.)	*y* (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00





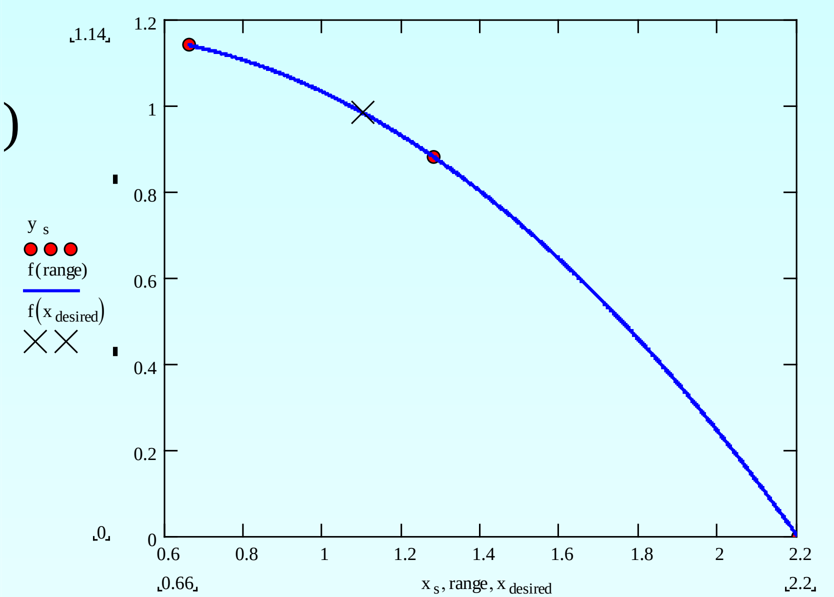
Quadratic Interpolation (contd)

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$x_0 = 2.20, \; y(x_0) = 0.00$$

$$x_1 = 1.28, \; y(x_1) = 0.88$$

$$x_2 = 0.66, \; y(x_2) = 1.14$$



Quadratic Interpolation (contd)

$$b_0 = y(x_0)$$

$$= 0.00$$

$$b_1 = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{0.88 - 0.00}{1.28 - 2.20}$$

$$= -0.95652$$

$$b_2 = \frac{\dfrac{y(x_2) - y(x_1)}{x_2 - x_1} - \dfrac{y(x_1) - y(x_0)}{x_1 - x_0}}{x_2 - x_0} = \frac{\dfrac{1.14 - 0.88}{0.66 - 1.28} - \dfrac{0.88 - 0.00}{1.28 - 2.20}}{0.66 - 2.20}$$

$$= \frac{-0.41935 + 0.95652}{-1.54}$$

$$= -0.34881$$

Quadratic Interpolation (contd)

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$= 0 - 0.95652(x - 2.20) - 0.34881(x - 2.20)(x - 1.28), \quad 0.66 \le x \le 2.20$$

At $x = 1.10$,

$$y(1.10) = 0 - 0.95652(1.10 - 2.20) - 0.34881(1.10 - 2.20)(1.10 - 1.28)$$

$$= 0.98311 \; in.$$

The absolute relative approximate error $\left|\in_a\right|$ obtained between the results from the first and second order polynomial is

$$\left|\in_a\right| = \left|\frac{0.98311 - 0.95548}{0.98311}\right| \times 100$$

$$= 2.8100\%$$

General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

General Form

Given $(n+1)$ data points, $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

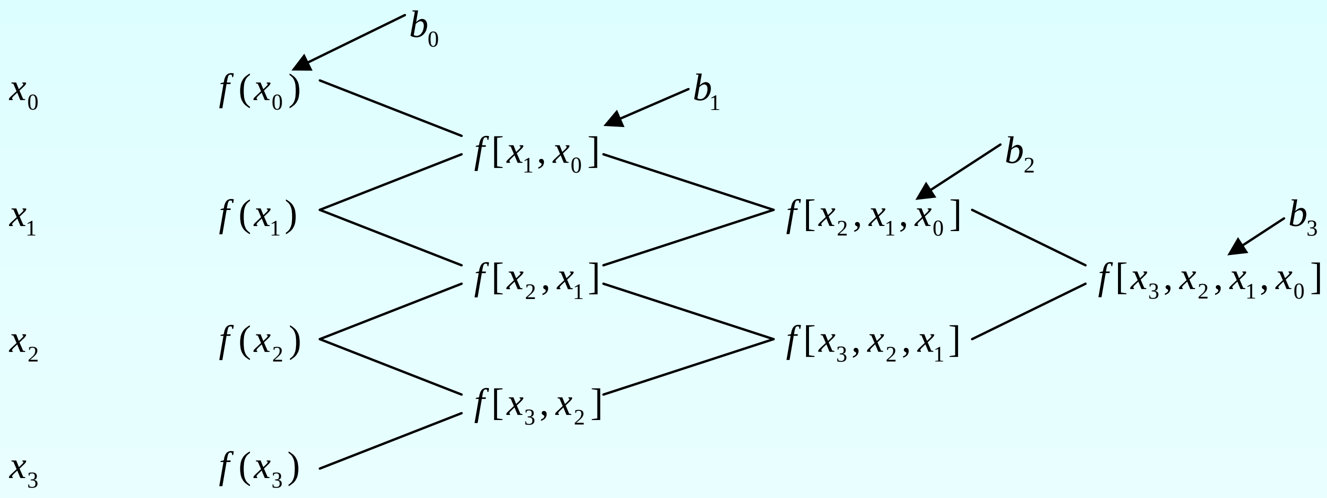
$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots., x_0]$$

$$b_n = f[x_n, x_{n-1}, \ldots., x_0]$$

General form

The third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3), is

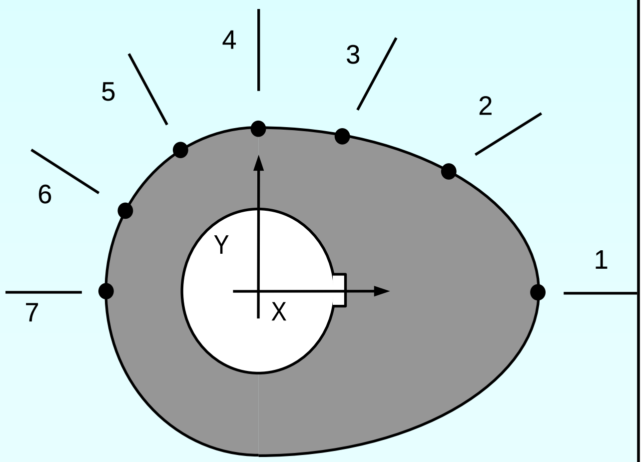
$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) \\ + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

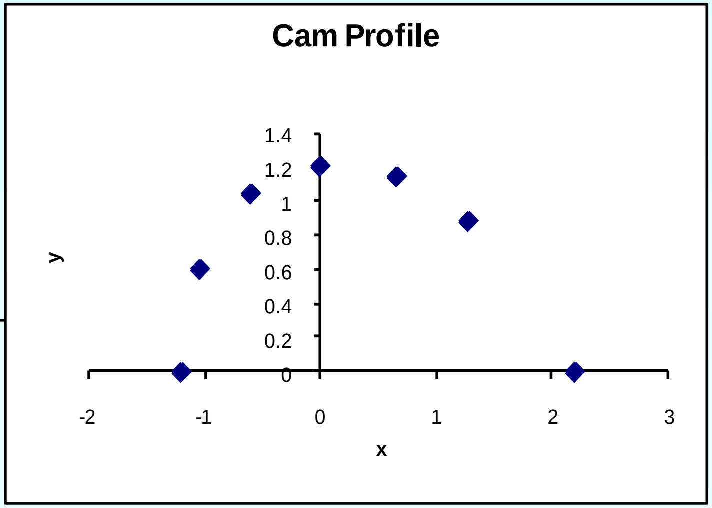


Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a straight line profile between x = 1.28 to x = 0.66, what is the value of y at x=1.1 ? Find using the Newton's divided difference method for a sixth order polynomial.

Point	*x* (in.)	*y* (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00





Example

The value of y profile is chosen as

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$
$$+ b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) + b_5(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$
$$+ b_6(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)$$

$x_0 = 2.20, \quad y(x_0) = 0.00$

$x_1 = 1.28, \quad y(x_1) = 0.88$

$x_2 = 0.66, \quad y(x_2) = 1.14$

$x_3 = 0.00, \quad y(x_3) = 1.20$

$x_4 = -0.60, \quad y(x_4) = 1.04$

$x_5 = -1.04, \quad y(x_5) = 0.60$

$x_6 = -1.20, \quad y(x_6) = 0.00$

Example

The values of the constants are found as:

$$b_0 = 0.00$$

$$b_1 = -0.95652$$

$$b_2 = -0.34881$$

$$b_3 = -0.041914$$

$$b_4 = -0.020135$$

$$b_5 = 0.024834$$

$$b_6 = -0.17103$$

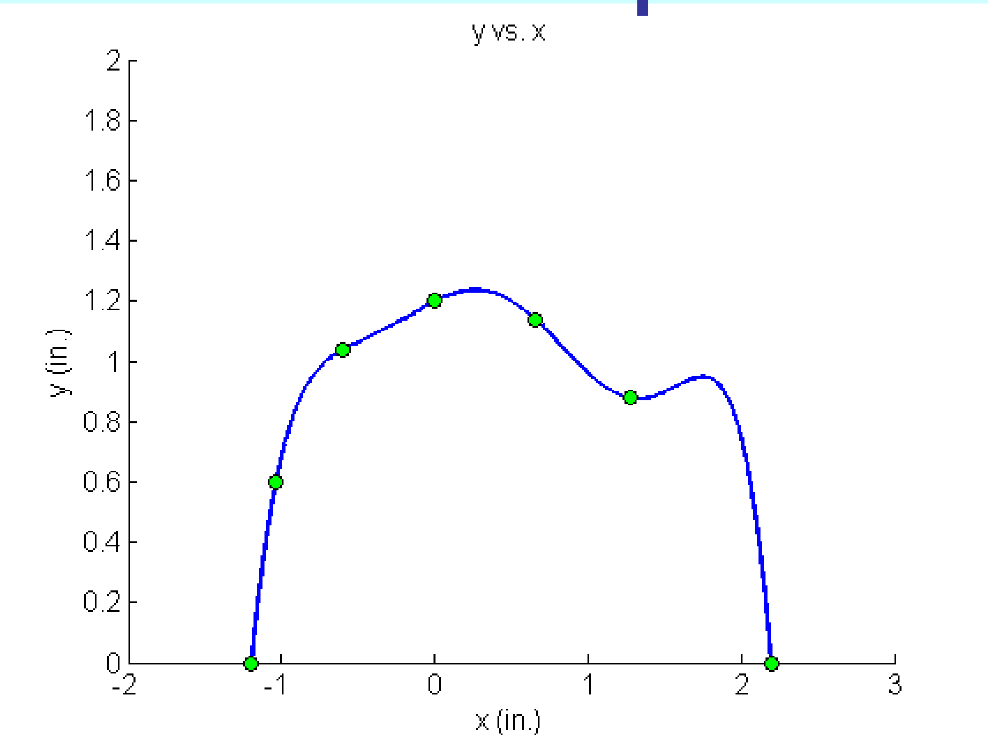
Example

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$
$$+ b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) + b_5(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$
$$+ b_6(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)$$

$$= 0 - 0.95652(x - 2.2) - 0.34881(x - 2.2)(x - 1.28)$$
$$- 0.041914(x - 2.2)(x - 1.28)(x - 0.66)$$
$$- 0.020135(x - 2.2)(x - 1.28)(x - 0.66)(x - 0)$$
$$+ 0.024834(x - 2.2)(x - 1.28)(x - 0.66)(x - 0)(x + 0.6)$$
$$- 0.17103(x - 2.2)(x - 1.28)(x - 0.66)(x - 0)(x + 0.6)(x + 1.04)$$

$$y(x) = 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3$$
$$+ 0.072013x^4 + 0.45241x^5 - 0.17103x^6, \quad -1.20 \le x \le 2.20$$

Example



$$y(x) = 0 - 0.95652(x-2.2) - 0.34881(x-2.2)(x-1.28) - 0.041914(x-2.2)(x-1.28)(x-0.66)$$
$$-0.020135(x-2.2)(x-1.28)(x-0.66)(x-0)$$
$$+0.024834(x-2.2)(x-1.28)(x-0.66)(x-0)(x+0.6)$$
$$-0.17103(x-2.2)(x-1.28)(x-0.66)(x-0)(x+0.6)(x+1.04)$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/newton_divided_difference_method.html

THE END

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