

## Chapter 05.02

### Direct Method of Interpolation

After reading this chapter, you should be able to:

1. apply the direct method of interpolation,
2. solve problems using the direct method of interpolation, and
3. use the direct method interpolants to find derivatives and integrals of discrete functions.

#### What is interpolation?

Many times, data is given only at discrete points such as  $(x_0, y_0)$ ,  $(x_1, y_1)$ , ...,  $(x_{n-1}, y_{n-1})$ ,  $(x_n, y_n)$ . So, how then does one find the value of  $y$  at any other value of  $x$ ? Well, a continuous function  $f(x)$  may be used to represent the  $n+1$  data values with  $f(x)$  passing through the  $n+1$  points (Figure 1). Then one can find the value of  $y$  at any other value of  $x$ . This is called *interpolation*.

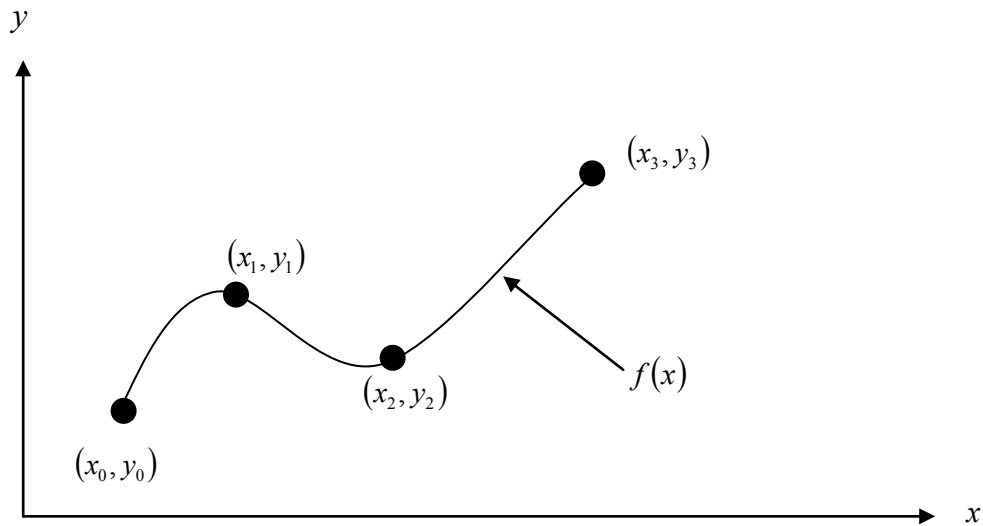
Of course, if  $x$  falls outside the range of  $x$  for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function  $f(x)$  should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

- (A) evaluate,
- (B) differentiate, and
- (C) integrate

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order  $n$  that passes through the  $n+1$  points. One of the methods of interpolation is called the direct method. Other methods include Newton's divided difference polynomial method and the Lagrangian interpolation method. We will discuss the direct method in this chapter.



**Figure 1** Interpolation of discrete data.

### Direct Method

The direct method of interpolation is based on the following premise. Given  $n + 1$  data points, fit a polynomial of order  $n$  as given below

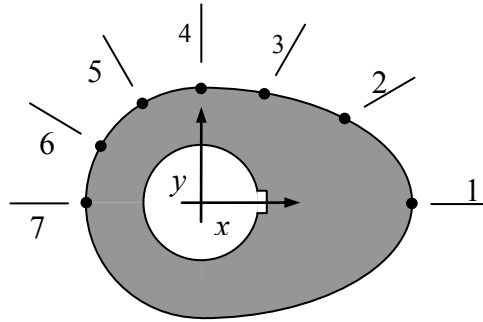
$$y = a_0 + a_1x + \dots + a_nx^n \quad (1)$$

through the data, where  $a_0, a_1, \dots, a_n$  are  $n + 1$  real constants. Since  $n + 1$  values of  $y$  are given at  $n + 1$  values of  $x$ , one can write  $n + 1$  equations. Then the  $n + 1$  constants,  $a_0, a_1, \dots, a_n$  can be found by solving the  $n + 1$  simultaneous linear equations. To find the value of  $y$  at a given value of  $x$ , simply substitute the value of  $x$  in Equation 1.

But, it is not necessary to use all the data points. How does one then choose the order of the polynomial and what data points to use? This concept and the direct method of interpolation are best illustrated using examples.

### Example 1

The geometry of a cam is given in Figure 2. A curve needs to be fit through the seven points given in Table 1 to fabricate the cam.



**Figure 2** Schematic of cam profile.

**Table 1** Geometry of the cam.

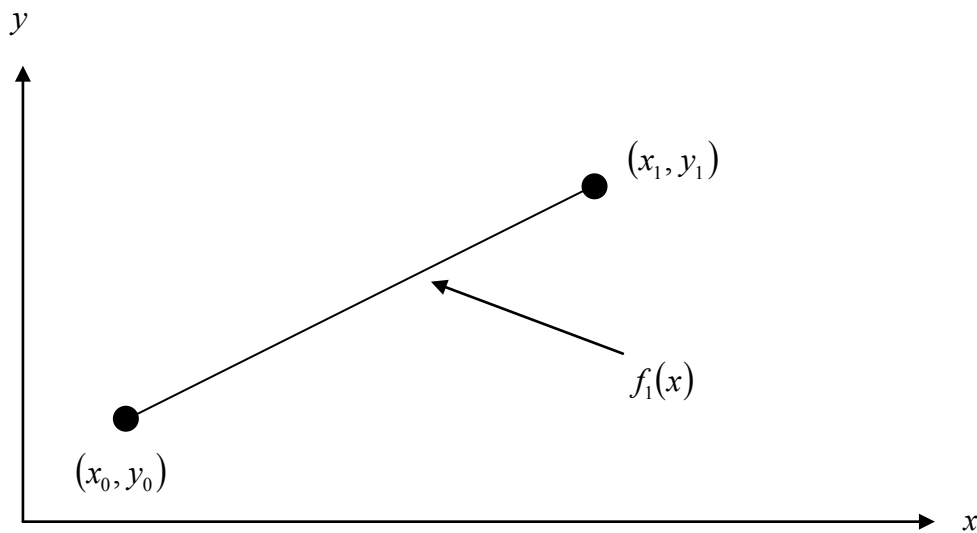
Point	$x$ (in.)	$y$ (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	-0.60	1.04
6	-1.04	0.60
7	-1.20	0.00

If the cam follows a straight line profile from  $x = 1.28$  to  $x = 0.66$ , what is the value of  $y$  at  $x = 1.10$  using the direct method of interpolation and a first order polynomial?

**Solution**

For first order polynomial interpolation (also called linear interpolation), we choose the value of  $y$  given by

$$y(x) = a_0 + a_1x$$



**Figure 3** Linear interpolation.

Since we want to find the value of  $y$  at  $x = 1.10$ , and we are using a first order polynomial, using the two points  $x_0 = 1.28$  and  $x_1 = 0.66$ , then

$$x_0 = 1.28, y(x_0) = 0.88$$

$$x_1 = 0.66, y(x_1) = 1.14$$

gives

$$y(1.28) = a_0 + a_1(1.28) = 0.88$$

$$y(0.66) = a_0 + a_1(0.66) = 1.14$$

Writing the equations in matrix form, we have

$$\begin{bmatrix} 1 & 1.28 \\ 1 & 0.66 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0.88 \\ 1.14 \end{bmatrix}$$

Solving the above two equations gives,

$$a_0 = 1.4168$$

$$a_1 = -0.41935$$

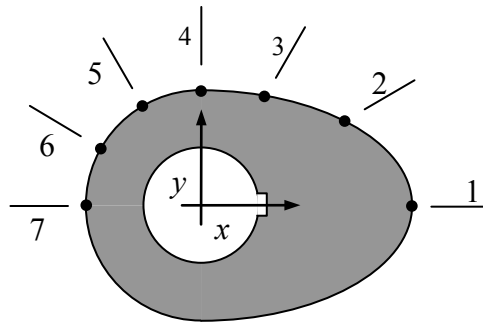
Hence

$$\begin{aligned} y(x) &= a_0 + a_1x \\ &= 1.4168 - 0.41935x, \quad 0.66 \leq x \leq 1.28 \end{aligned}$$

$$\begin{aligned} y(1.10) &= 1.4168 - 0.41935(1.10) \\ &= 0.95548 \text{ in.} \end{aligned}$$

### Example 2

The geometry of a cam is given in Figure 4. A curve needs to be fit through the seven points given in Table 2 to fabricate the cam.



**Figure 4** Schematic of cam profile.

**Table 2** Geometry of the cam.

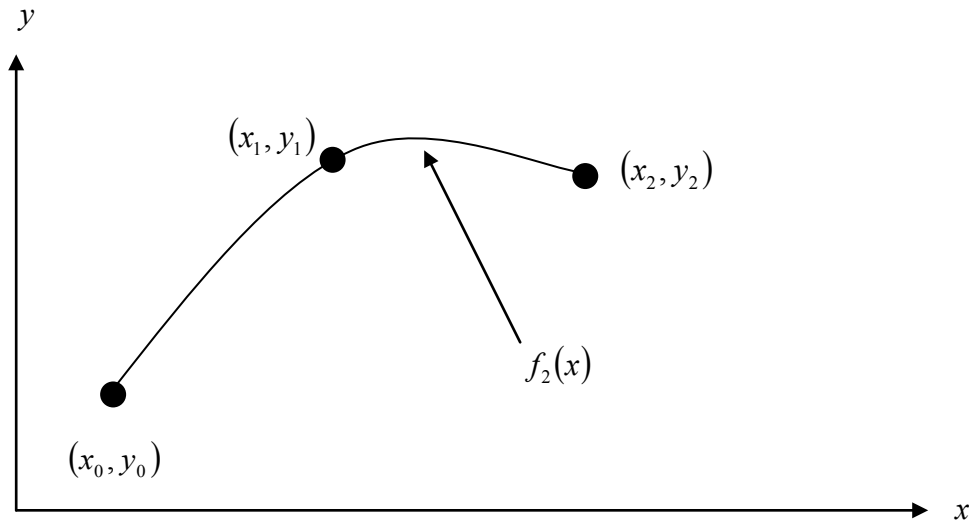
Point	$x$ (in.)	$y$ (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	-0.60	1.04
6	-1.04	0.60
7	-1.20	0.00

If the cam follows a quadratic profile from  $x = 2.20$  to  $x = 1.28$  to  $x = 0.66$ , what is the value of  $y$  at  $x = 1.10$  using the direct method of interpolation and a second order polynomial? Find the absolute relative approximate error for the second order polynomial approximation.

### Solution

For second order polynomial interpolation (also called quadratic interpolation), we choose the value of  $y$  given by

$$y(x) = a_0 + a_1x + a_2x^2$$



**Figure 5** Quadratic interpolation.

Since we want to find the value of  $y$  at  $x=1.10$ , and we are using a second order polynomial, using the three points  $x_0 = 2.20$ ,  $x_1 = 1.28$  and  $x_2 = 0.66$ , then

$$x_0 = 2.20, \quad y(x_0) = 0.00$$

$$x_1 = 1.28, \quad y(x_1) = 0.88$$

$$x_2 = 0.66, \quad y(x_2) = 1.14$$

gives

$$y(2.20) = a_0 + a_1(2.20) + a_2(2.20)^2 = 0.00$$

$$y(1.28) = a_0 + a_1(1.28) + a_2(1.28)^2 = 0.88$$

$$y(0.66) = a_0 + a_1(0.66) + a_2(0.66)^2 = 1.14$$

Writing the three equations in matrix form, we have

$$\begin{bmatrix} 1 & 2.20 & 4.84 \\ 1 & 1.28 & 1.6384 \\ 1 & 0.66 & 0.4356 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.66 \\ 1.14 \end{bmatrix}$$

Solving the above three equations gives

$$a_0 = 1.1221$$

$$a_1 = 0.25734$$

$$a_2 = -0.34881$$

Hence

$$y(x) = 1.1221 + 0.25734x - 0.34881x^2, \quad 0.66 \leq x \leq 2.20$$

At  $x = 1.10$ ,

$$\begin{aligned} y(1.10) &= 1.1221 + 0.25734(1.10) - 0.34881(1.10)^2 \\ &= 0.98311 \text{ in} \end{aligned}$$

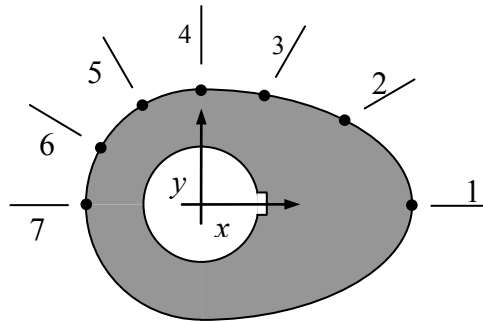
The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the first and second order polynomial is

$$|\epsilon_a| = \left| \frac{0.98311 - 0.95548}{0.98311} \right| \times 100$$

$$= 2.8100\%$$

### Example 3

The geometry of a cam is given in Figure 6. A curve needs to be fit through the seven points given in Table 3 to fabricate the cam.



**Figure 6** Schematic of cam profile.

**Table 3** Geometry of the cam.

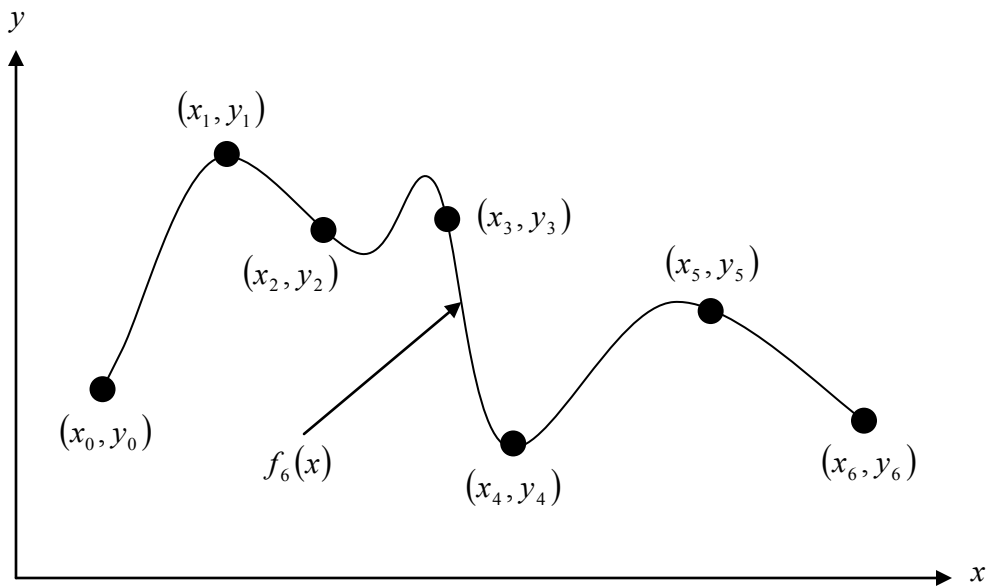
Point	$x$ (in.)	$y$ (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	-0.60	1.04
6	-1.04	0.60
7	-1.20	0.00

Find the cam profile using all seven points in Table 3 using the direct method of interpolation and a sixth order polynomial.

### Solution

For the sixth order polynomial, we choose the value of  $y$  given by

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$$



**Figure 7** 6<sup>th</sup> order polynomial interpolation.

Using the seven points,

$$x_0 = 2.20, \quad y(x_0) = 0$$

$$x_1 = 1.28, \quad y(x_1) = 0.88$$

$$x_2 = 0.66, \quad y(x_2) = 1.14$$

$$x_3 = 0.00, \quad y(x_3) = 1.20$$

$$x_4 = -0.60, \quad y(x_4) = 1.04$$

$$x_5 = -1.04, \quad y(x_5) = 0.60$$

$$x_6 = -1.20, \quad y(x_6) = 0$$

gives

$$y(2.20) = 0.00 = a_0 + a_1(2.20) + a_2(2.20)^2 + a_3(2.20)^3 + a_4(2.20)^4 + a_5(2.20)^5 + a_6(2.20)^6$$

$$y(1.28) = 0.88 = a_0 + a_1(1.28) + a_2(1.28)^2 + a_3(1.28)^3 + a_4(1.28)^4 + a_5(1.28)^5 + a_6(1.28)^6$$

$$y(0.66) = 1.14 = a_0 + a_1(0.66) + a_2(0.66)^2 + a_3(0.66)^3 + a_4(0.66)^4 + a_5(0.66)^5 + a_6(0.66)^6$$

$$y(0.00) = 1.20 = a_0 + a_1(0.00) + a_2(0.00)^2 + a_3(0.00)^3 + a_4(0.00)^4 + a_5(0.00)^5 + a_6(0.00)^6$$



$$y(-0.60) = 1.04 = a_0 + a_1(-0.60) + a_2(-0.60)^2 + a_3(-0.60)^3 + a_4(-0.60)^4 + a_5(-0.60)^5 + a_6(-0.60)^6$$

$$y(-1.04) = 0.60 = a_0 + a_1(-1.04) + a_2(-1.04)^2 + a_3(-1.04)^3 + a_4(-1.04)^4 + a_5(-1.04)^5 + a_6(-1.04)^6$$

$$y(-1.20) = 0.00 = a_0 + a_1(-1.20) + a_2(-1.20)^2 + a_3(-1.20)^3 + a_4(-1.20)^4 + a_5(-1.20)^5 + a_6(-1.20)^6$$

Writing the seven equations in matrix form, we have

$$\begin{bmatrix} 1 & 2.20 & 2.20^2 & 2.20^3 & 2.20^4 & 2.20^5 & 2.20^6 \\ 1 & 1.28 & 1.28^2 & 1.28^3 & 1.28^4 & 1.28^5 & 1.28^6 \\ 1 & 0.66 & 0.66^2 & 0.66^3 & 0.66^4 & 0.66^5 & 0.66^6 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -0.60 & 0.60^2 & -0.60^3 & 0.60^4 & -0.60^5 & 0.60^6 \\ 1 & -1.04 & 1.04^2 & -1.04^3 & 1.04^4 & -1.04^5 & 1.04^6 \\ 1 & -1.20 & 1.20^2 & -1.20^3 & 1.20^4 & -1.20^5 & 1.20^6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.88 \\ 1.14 \\ 1.20 \\ 1.04 \\ 0.60 \\ 0.00 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2.20 & 4.84 & 10.648 & 23.426 & 51.536 & 113.38 \\ 1 & 1.28 & 1.6384 & 2.0972 & 2.6844 & 3.4360 & 4.3980 \\ 1 & 0.66 & 0.4356 & 0.28750 & 0.18975 & 0.12523 & 0.082654 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -0.60 & 0.36 & -0.216 & 0.1296 & -0.07776 & 0.046656 \\ 1 & -1.04 & 1.0816 & -1.1249 & 1.1699 & -1.2167 & 1.2653 \\ 1 & -1.20 & 1.44 & -1.728 & 2.0736 & -2.4883 & 2.9860 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.88 \\ 1.14 \\ 1.20 \\ 1.04 \\ 0.60 \\ 0.00 \end{bmatrix}$$

Solving the above seven equations gives

$$a_0 = 1.2$$

$$a_1 = 0.25112$$

$$a_2 = -0.27255$$

$$a_3 = -0.56765$$

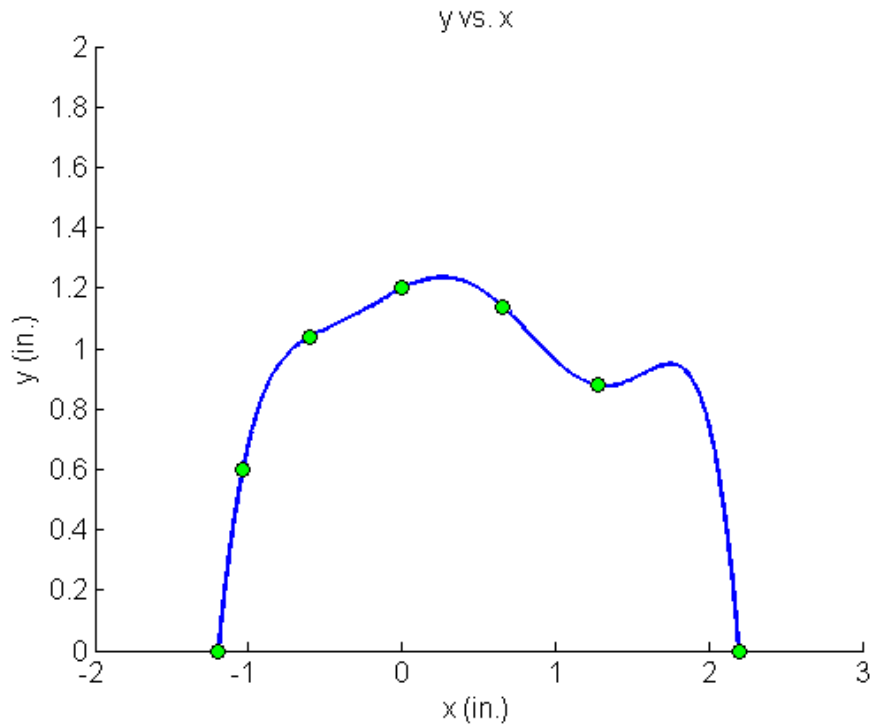
$$a_4 = 0.072013$$

$$a_5 = 0.45241$$

$$a_6 = -0.17103$$

Hence

$$\begin{aligned} y(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \\ &= 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3 \\ &\quad + 0.072013x^4 + 0.45241x^5 - 0.17103x^6, \quad -1.20 \leq x \leq 2.20 \end{aligned}$$



**Figure 8** Plot of the cam profile as defined by a 6<sup>th</sup> order interpolating polynomial (using directed method of interpolation).

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#### INTERPOLATION

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Topic	Direct Method of Interpolation
Summary	Textbook notes on the direct method of interpolation.
Major	Industrial Engineering
Authors	Autar Kaw, Peter Warr, Michael Keteltas
Date	November 15, 2012
Web Site	<a href="http://numericalmethods.eng.usf.edu">http://numericalmethods.eng.usf.edu</a>

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