Chapter 05.04
Lagrangian Interpolation

After reading this chapter, you should be able to:
1. *derive Lagrangian method of interpolation,*
2. *solve problems using Lagrangian method of interpolation, and*
3. *use Lagrangian interpolants to find derivatives and integrals of discrete functions.*

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

(A) evaluate,
(B) differentiate, and
(C) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ data points. One of the methods used to find this polynomial is called the Lagrangian method of interpolation. Other methods include Newton's divided difference polynomial method and the direct method. We discuss the Lagrangian method in this chapter.

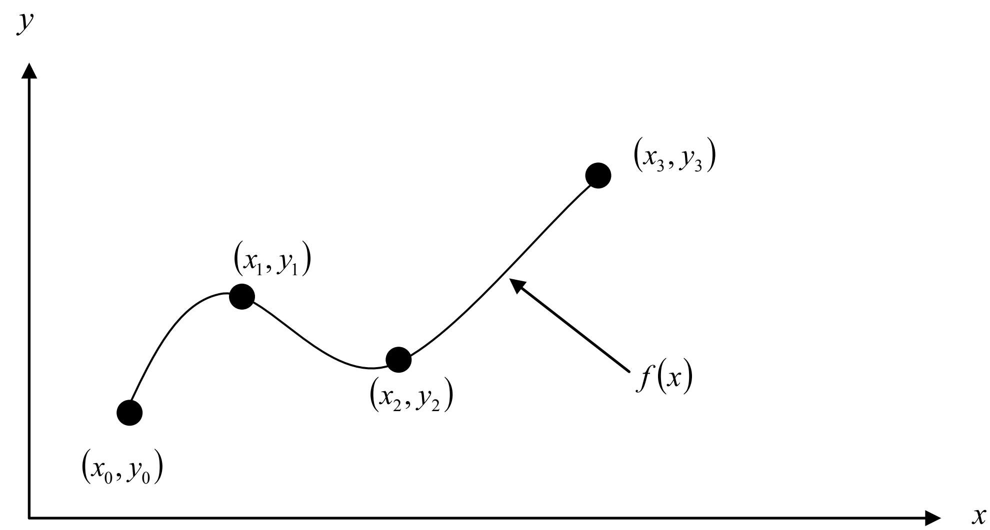

Figure 1 Interpolation of discrete data.

The Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $n+1$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j = i$ omitted. The application of Lagrangian interpolation will be clarified using an example.

Example 1

The geometry of a cam is given in Figure 2. A curve needs to be fit through the seven points given in Table 1 to fabricate the cam.

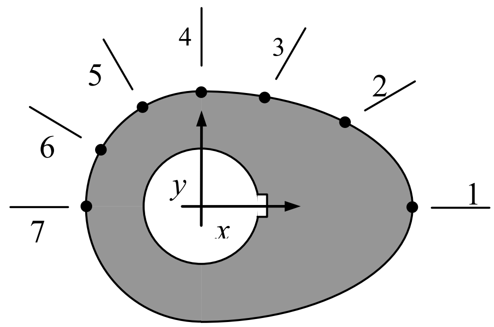

Figure 2 Schematic of cam profile.

Table 1 Geometry of the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00

If the cam follows a straight line profile from $x = 1.28$ to $x = 0.66$, what is the value of y at $x = 1.10$ using a first order Lagrange polynomial?

Solution

For first order Lagrange polynomial interpolation (also called linear interpolation), the value of y is given by

$$\begin{aligned} y(x) &= \sum_{i=0}^{1} L_i(x) y(x_i) \\ &= L_0(x) y(x_0) + L_1(x) y(x_1) \end{aligned}$$

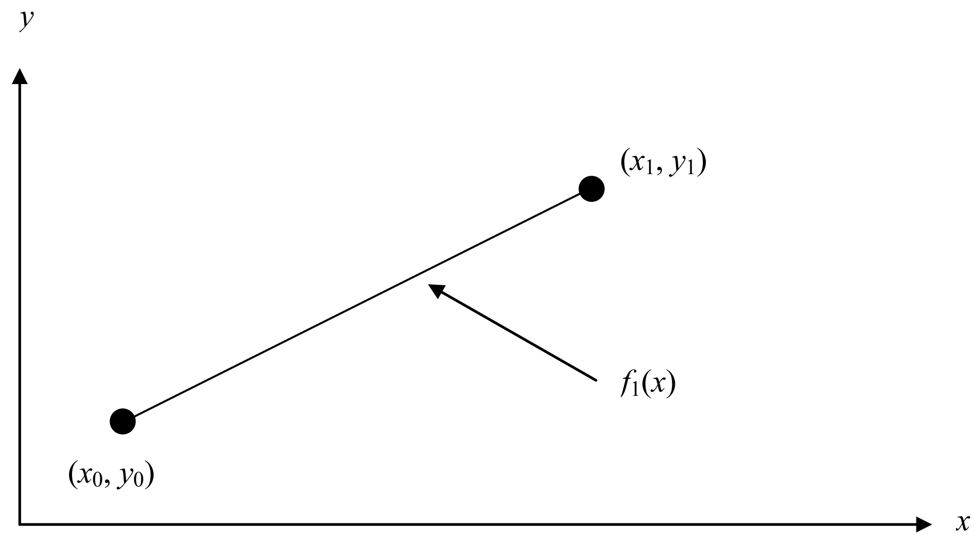

Figure 3 Linear interpolation.

Since we want to find the value of y at $x = 1.10$, using the two points $x_0 = 1.28$ and $x_1 = 0.66$, then

$$x_0 = 1.28,\ y(x_0) = 0.88$$
$$x_1 = 0.66,\ y(x_1) = 1.14$$

gives

$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{x - x_j}{x_0 - x_j}$$
$$= \frac{x - x_1}{x_0 - x_1}$$
$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^{1} \frac{x - x_j}{x_1 - x_j}$$
$$= \frac{x - x_0}{x_1 - x_0}$$

Hence

$$y(x) = \frac{x - x_1}{x_0 - x_1} y(x_0) + \frac{x - x_0}{x_1 - x_0} y(x_1)$$
$$= \frac{x - 0.66}{1.28 - 0.66}(0.88) + \frac{x - 1.28}{0.66 - 1.28}(1.14), \quad 0.66 \le x \le 1.28$$
$$y(1.10) = \frac{1.10 - 0.66}{1.28 - 0.66}(0.88) + \frac{1.10 - 1.28}{0.66 - 1.28}(1.14)$$
$$= 0.70968(0.88) + 0.29032(1.14)$$
$$= 0.95548 \text{ in.}$$

You can see that $L_0(x) = 0.70968$ and $L_1(x) = 0.29032$ are like weightages given to the values of y at $x = 1.28$ and $x = 0.66$ to calculate the value of y at $x = 1.10$.

Quadratic Interpolation

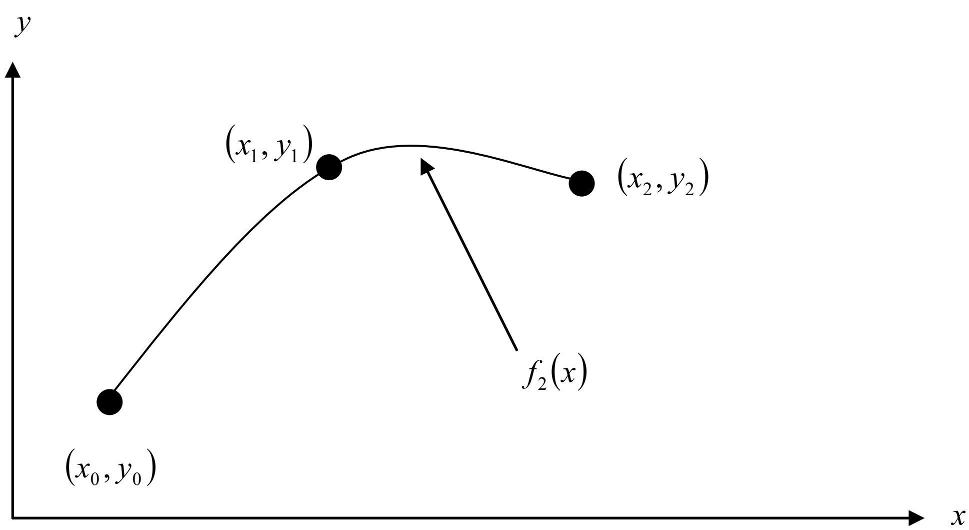

Figure 4 Quadratic interpolation.

Example 2

The geometry of a cam is given in Figure 5. A curve needs to be fit through the seven points given in Table 2 to fabricate the cam.

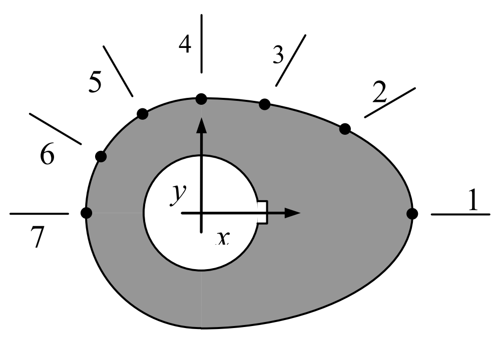

Figure 5 Schematic of cam profile.

Table 2 Geometry of the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00

If the cam follows a quadratic profile from $x = 2.20$ to $x = 1.28$ to $x = 0.66$, what is the value of y at $x = 1.10$ using a second order Lagrange polynomial? Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For second order Lagrange polynomial interpolation (also called quadratic interpolation), the value of y given by

$$\begin{aligned} y(x) &= \sum_{i=0}^{2} L_i(x) y(x_i) \\ &= L_0(x)y(x_0) + L_1(x)y(x_1) + L_2(x)y(x_2) \end{aligned}$$

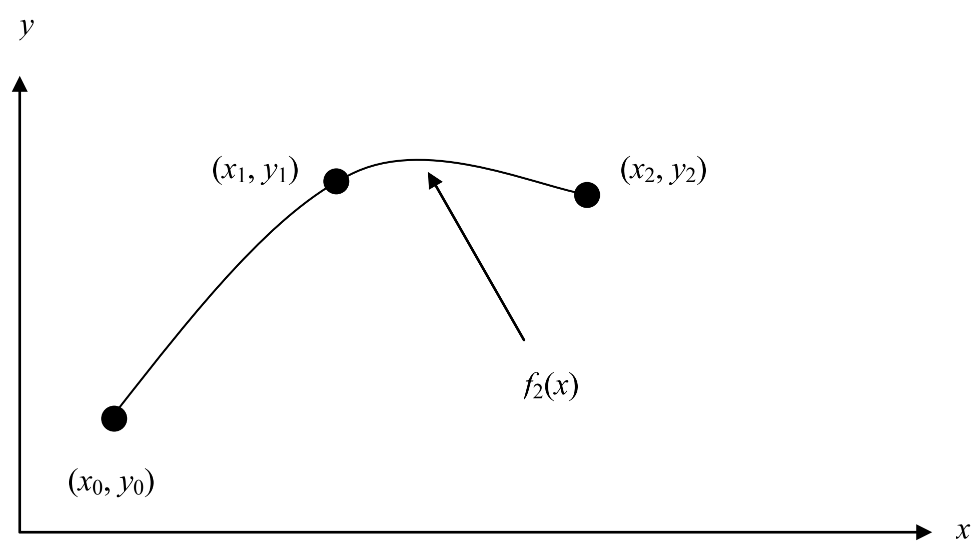

Figure 6 Quadratic interpolation.

Since we want to find the value of y at $x = 1.10$, using the three points $x_0 = 2.20$, $x_1 = 1.28$, $x_2 = 0.66$, then

$$x_0 = 2.20, \quad y(x_0) = 0.00$$
$$x_1 = 1.28, \quad y(x_1) = 0.88$$
$$x_2 = 0.66, \quad y(x_2) = 1.14$$

gives

$$L_0(x)=\prod_{\substack{j=0\\j\neq 0}}^{2}\frac{x-x_j}{x_0-x_j}$$

$$=\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)$$

$$L_1(x)=\prod_{\substack{j=0\\j\neq 1}}^{2}\frac{x-x_j}{x_1-x_j}$$

$$=\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)$$

$$L_2(x)=\prod_{\substack{j=0\\j\neq 2}}^{2}\frac{x-x_j}{x_2-x_j}$$

$$=\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)$$

Hence

$$y(x)=\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)y(x_0)+\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)y(x_1)+\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)y(x_2),$$

$$x_0\le x\le x_2$$

$$y(1.10)=\frac{(1.10-1.28)(1.10-0.66)}{(2.20-1.28)(2.20-0.66)}(0.00)+\frac{(1.10-2.20)(1.10-0.66)}{(1.28-2.20)(1.28-0.66)}(0.88)$$

$$+\frac{(1.10-2.20)(1.10-1.28)}{(0.66-2.20)(0.66-1.28)}(1.14)$$

$$=(-0.055901)(0.00)+(0.84853)(0.88)+(0.20737)(1.14)$$

$$=0.98311\text{ in.}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a|=\left|\frac{0.98311-0.95548}{0.98311}\right|\times 100$$

$$=2.8100\%$$

Example 3

The geometry of a cam is given in Figure 7. A curve needs to be fit through the seven points given in Table 3 to fabricate the cam.

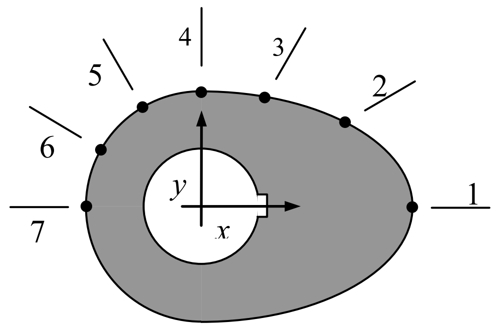

Figure 7 Schematic of cam profile.

Table 3 Geometry of the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00

Find the cam profile using all seven points in Table 3 and a sixth order Lagrange polynomial.

Solution

For the sixth order polynomial, we choose the value of y given by

$$\begin{aligned} y(x) &= \sum_{i=0}^{6} L_i(x)y(x_i) \\ &= L_0(x)y(x_0) + L_1(x)y(x_1) + L_2(x)y(x_2) + L_3(x)y(x_3) \\ &\qquad + L_4(x)y(x_4) + L_5(x)y(x_5) + L_6(x)y(x_6) \end{aligned}$$

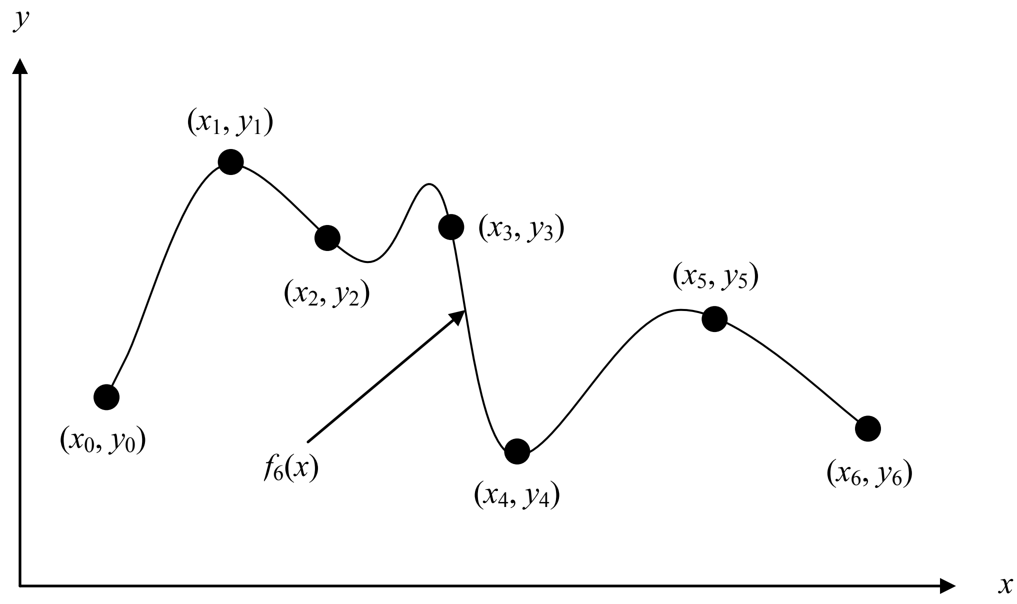

Figure 8 6^{th} order polynomial interpolation.

Using the seven points,

$$x_0 = 2.20,\ \ y(x_0) = 0.00$$
$$x_1 = 1.28,\ \ y(x_1) = 0.88$$
$$x_2 = 0.66,\ \ y(x_2) = 1.14$$
$$x_3 = 0.00,\ \ y(x_3) = 1.20$$
$$x_4 = -.60,\ \ y(x_4) = 1.04$$
$$x_5 = -1.04,\ \ y(x_5) = 0.60$$
$$x_6 = -1.20,\ \ y(x_6) = 0.00$$

gives

$$L_0(x) = \prod_{\substack{j=0 \\ j\neq 0}}^{6} \frac{x - x_j}{x_0 - x_j}$$
$$= \left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)\left(\frac{x-x_3}{x_0-x_3}\right)\left(\frac{x-x_4}{x_0-x_4}\right)\left(\frac{x-x_5}{x_0-x_5}\right)\left(\frac{x-x_6}{x_0-x_6}\right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j\neq 1}}^{6} \frac{x - x_j}{x_1 - x_j}$$
$$= \left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)\left(\frac{x-x_4}{x_1-x_4}\right)\left(\frac{x-x_5}{x_1-x_5}\right)\left(\frac{x-x_6}{x_1-x_6}\right)$$

$$L_2(x) = \prod_{\substack{j=0 \\ j\neq 2}}^{6} \frac{x - x_j}{x_2 - x_j}$$

$$= \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_3}{x_2 - x_3}\right)\left(\frac{x - x_4}{x_2 - x_4}\right)\left(\frac{x - x_5}{x_2 - x_5}\right)\left(\frac{x - x_6}{x_2 - x_6}\right)$$

$$L_3(x) = \prod_{\substack{j=0 \\ j \neq 3}}^{6} \frac{x - x_j}{x_3 - x_j}$$

$$= \left(\frac{x - x_0}{x_3 - x_0}\right)\left(\frac{x - x_1}{x_3 - x_1}\right)\left(\frac{x - x_2}{x_3 - x_2}\right)\left(\frac{x - x_4}{x_3 - x_4}\right)\left(\frac{x - x_5}{x_3 - x_5}\right)\left(\frac{x - x_6}{x_3 - x_6}\right)$$

$$L_4(x) = \prod_{\substack{j=0 \\ j \neq 4}}^{6} \frac{x - x_j}{x_4 - x_j}$$

$$= \left(\frac{x - x_0}{x_4 - x_0}\right)\left(\frac{x - x_1}{x_4 - x_1}\right)\left(\frac{x - x_2}{x_4 - x_2}\right)\left(\frac{x - x_3}{x_4 - x_3}\right)\left(\frac{x - x_5}{x_4 - x_5}\right)\left(\frac{x - x_6}{x_4 - x_6}\right)$$

$$L_5(x) = \prod_{\substack{j=0 \\ j \neq 5}}^{6} \frac{x - x_j}{x_5 - x_j}$$

$$= \left(\frac{x - x_0}{x_5 - x_0}\right)\left(\frac{x - x_1}{x_5 - x_1}\right)\left(\frac{x - x_2}{x_5 - x_2}\right)\left(\frac{x - x_3}{x_5 - x_3}\right)\left(\frac{x - x_4}{x_5 - x_4}\right)\left(\frac{x - x_6}{x_5 - x_6}\right)$$

$$L_6(x) = \prod_{\substack{j=0 \\ j \neq 6}}^{6} \frac{x - x_j}{x_6 - x_j}$$

$$= \left(\frac{x - x_0}{x_6 - x_0}\right)\left(\frac{x - x_1}{x_6 - x_1}\right)\left(\frac{x - x_2}{x_6 - x_2}\right)\left(\frac{x - x_3}{x_6 - x_3}\right)\left(\frac{x - x_4}{x_6 - x_4}\right)\left(\frac{x - x_5}{x_6 - x_5}\right)$$

$$
\begin{aligned}
y(x) &= \left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)\left(\frac{x-x_3}{x_0-x_3}\right)\left(\frac{x-x_4}{x_0-x_4}\right)\left(\frac{x-x_5}{x_0-x_5}\right)\left(\frac{x-x_6}{x_0-x_6}\right)y(x_0) \\
&+ \left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)\left(\frac{x-x_4}{x_1-x_4}\right)\left(\frac{x-x_5}{x_1-x_5}\right)\left(\frac{x-x_6}{x_1-x_6}\right)y(x_1) \\
&+ \left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right)\left(\frac{x-x_4}{x_2-x_4}\right)\left(\frac{x-x_5}{x_2-x_5}\right)\left(\frac{x-x_6}{x_2-x_6}\right)y(x_2) \\
&+ \left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)\left(\frac{x-x_4}{x_3-x_4}\right)\left(\frac{x-x_5}{x_3-x_5}\right)\left(\frac{x-x_6}{x_3-x_6}\right)y(x_3) \\
&+ \left(\frac{x-x_0}{x_4-x_0}\right)\left(\frac{x-x_1}{x_4-x_1}\right)\left(\frac{x-x_2}{x_4-x_2}\right)\left(\frac{x-x_3}{x_4-x_3}\right)\left(\frac{x-x_5}{x_4-x_5}\right)\left(\frac{x-x_6}{x_4-x_6}\right)y(x_4) \\
&+ \left(\frac{x-x_0}{x_5-x_0}\right)\left(\frac{x-x_1}{x_5-x_1}\right)\left(\frac{x-x_2}{x_5-x_2}\right)\left(\frac{x-x_3}{x_5-x_3}\right)\left(\frac{x-x_4}{x_5-x_4}\right)\left(\frac{x-x_6}{x_5-x_6}\right)y(x_5) \\
&+ \left(\frac{x-x_0}{x_6-x_0}\right)\left(\frac{x-x_1}{x_6-x_1}\right)\left(\frac{x-x_2}{x_6-x_2}\right)\left(\frac{x-x_3}{x_6-x_3}\right)\left(\frac{x-x_4}{x_6-x_4}\right)\left(\frac{x-x_5}{x_6-x_5}\right)y(x_6)
\end{aligned}
$$

$$
\begin{aligned}
y(x) &= \frac{(x-1.28)(x-0.66)(x-0.00)(x+0.60)(x+1.04)(x+1.20)}{(2.20-1.28)(2.20-0.66)(2.20-0.00)(2.20+0.60)(2.20+1.04)(2.20+1.20)}(0.00) \\
&+ \frac{(x-2.20)(x-0.66)(x-0.00)(x+0.60)(x+1.04)(x+1.20)}{(1.28-2.20)(1.28-0.66)(1.28-0.00)(1.28+0.60)(1.28+1.04)(1.28+1.20)}(0.88) \\
&+ \frac{(x-2.20)(x-1.28)(x-0.00)(x+0.60)(x+1.04)(x+1.20)}{(0.66-2.20)(0.66-1.28)(0.66-0.00)(0.66+0.60)(0.66+1.04)(0.66+1.20)}(1.14) \\
&+ \frac{(x-2.20)(x-1.28)(x-0.66)(x+0.60)(x+1.04)(x+1.20)}{(0.00-2.20)(0.00-1.28)(0.00-0.66)(0.00+0.60)(0.00+1.04)(0.00+1.20)}(1.20) \\
&+ \frac{(x-2.20)(x-1.28)(x-0.66)(x-0.00)(x+1.04)(x+1.20)}{(-0.60-2.20)(-0.60-1.28)(-0.60-0.66)(-0.60-0.00)(-0.60+1.04)(-0.60+1.20)}(1.04) \\
&+ \frac{(x-2.20)(x-1.28)(x-0.66)(x-0.00)(x+0.60)(x+1.20)}{(-1.04-2.20)(-1.04-1.28)(-1.04-0.66)(-1.04-0.00)(-1.04+0.60)(-1.04+1.20)}(0.60) \\
&+ \frac{(x-2.20)(x-1.28)(x-0.66)(x-0.00)(x+0.60)(x+1.04)}{(-1.20-2.20)(-1.20-1.28)(-1.20-0.66)(-1.20-0.00)(-1.20+0.60)(-1.20+1.04)}(0.00)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{x^6 - 0.02x^5 - 4.0784x^4 - 2.5406x^3 + 1.6220x^2 + 1.0873x}{-8.9744} \\
&\quad + \frac{x^6 - 0.64x^5 - 4.4752x^4 - 0.27392x^3 + 4.6932x^2 + 2.1086x}{2.2023} \\
&\quad + \frac{x^6 - 1.3x^5 - 4.0528x^4 + 2.6797x^3 + 4.8740x^2 - 0.98892x - 1.3917}{-1.1597} \\
&\quad + \frac{x^6 - 1.9x^5 - 2.9128x^4 + 4.4274x^3 + 2.2176x^2 - 2.3195x}{1.0102} \\
&\quad + \frac{x^6 - 2.34x^5 - 1.6192x^4 + 4.3637x^3 + 0.33581x^2 - 1.3382x}{-1.5593}
\end{aligned}
$$

$$
\begin{aligned}
y(x) &= 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3 \\
&\quad + 0.072013x^4 + 0.45241x^5 + 0.17103x^6, \quad -1.20 \le x \le 2.20
\end{aligned}
$$

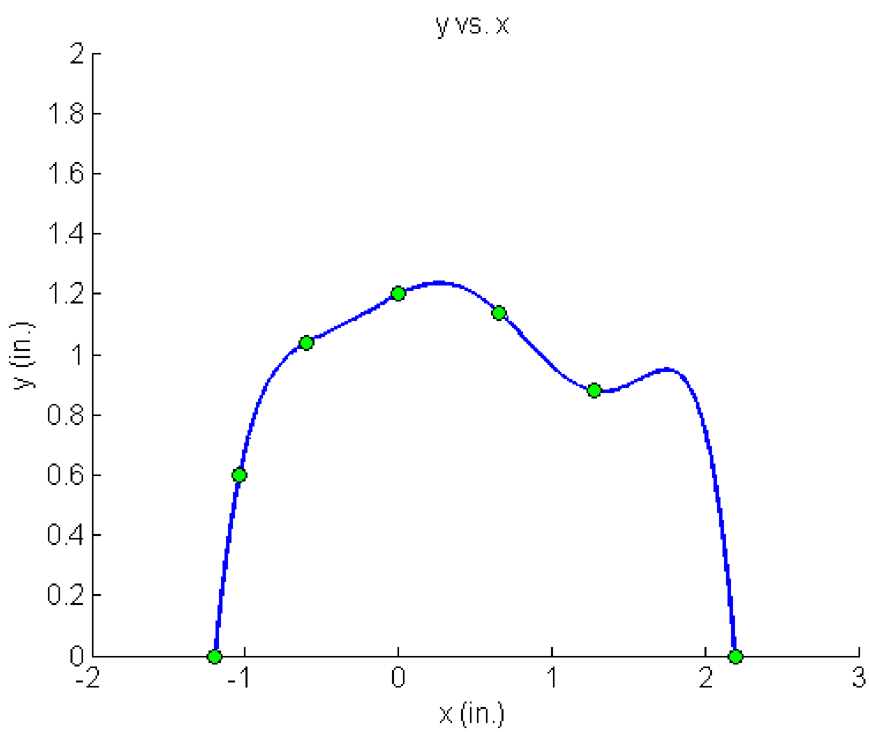

Figure 9 Plot of the cam profile as defined by a 6th order interpolating polynomial (using Lagrangian method of interpolation).

INTERPOLATION	
Topic	Lagrange Interpolation
Summary	Textbook notes on the Lagrangian method of interpolation
Major	Industrial Engineering
Authors	Autar Kaw, Michael Keteltas
Last Revised	November 15, 2012
Web Site	http://numericalmethods.eng.usf.edu