

# Linear Regression

Industrial Engineering Majors

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Transforming Numerical Methods Education for STEM  
Undergraduates

# Linear Regression

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# What is Regression?

What is regression? Given  $n$  data points  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  best fit  $y = f(x)$  to the data. The best fit is generally based on minimizing the sum of the square of the residuals,  $S_r$ .

Residual at a point is

$$\varepsilon_i = y_i - f(x_i)$$

Sum of the square of the residuals

$$S_r = \sum_{i=1}^n (y_i - f(x_i))^2$$

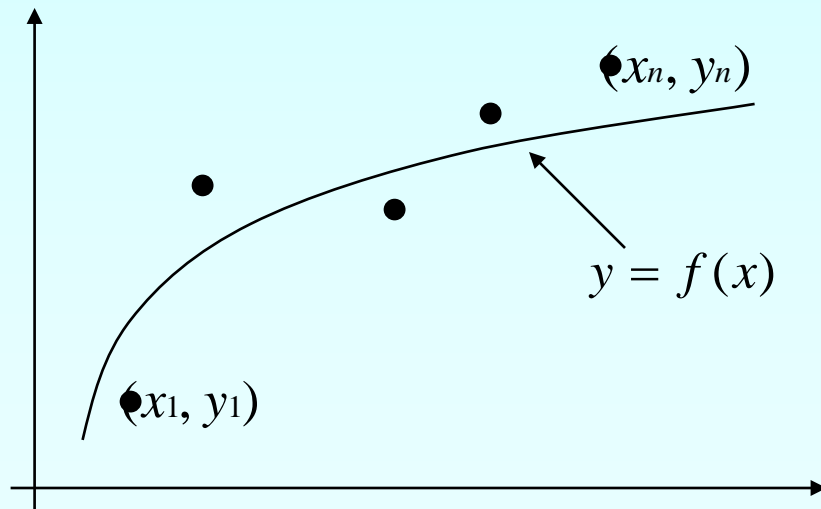
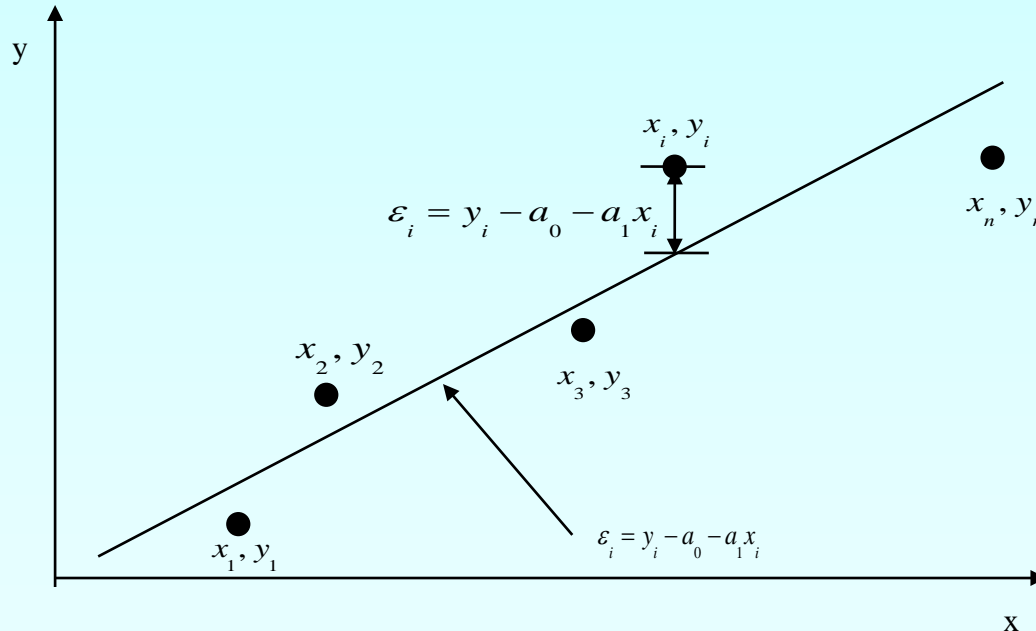


Figure. Basic model for regression

# Linear Regression-Criterion#1

Given  $n$  data points  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  best fit  $y = a_0 + a_1x$  to the data.



**Figure.** Linear regression of  $y$  vs.  $x$  data showing residuals at a typical point,  $x_i$ .

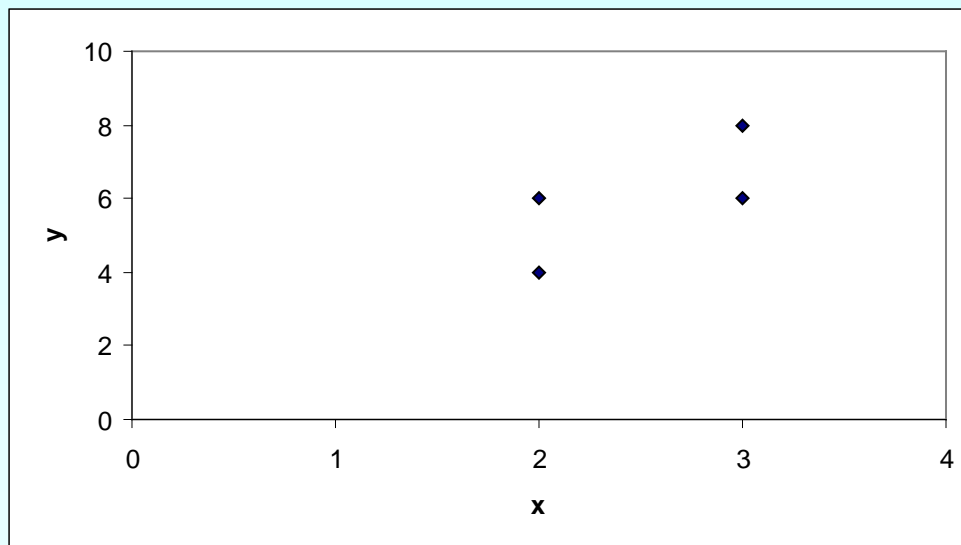
Does minimizing  $\sum_{i=1}^n \varepsilon_i$  work as a criterion, where  $\varepsilon_i = y_i - (a_0 + a_1x_i)$

# Example for Criterion#1

Example: Given the data points  $(2,4)$ ,  $(3,6)$ ,  $(2,6)$  and  $(3,8)$ , best fit the data to a straight line using Criterion#1

**Table.** Data Points

x	y
2.0	4.0
3.0	6.0
2.0	6.0
3.0	8.0



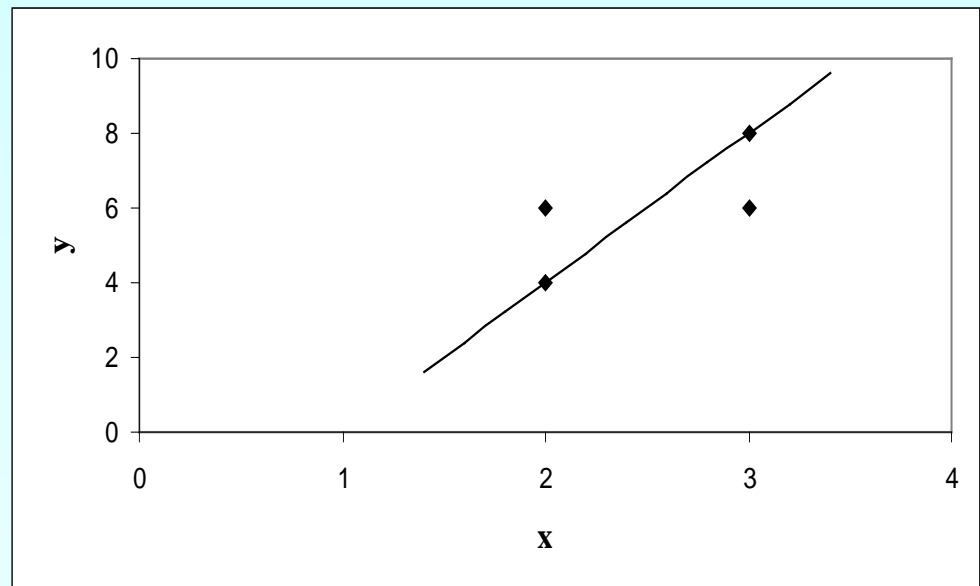
**Figure.** Data points for y vs. x data.

# Linear Regression-Criteria#1

Using  $y=4x-4$  as the regression curve

**Table.** Residuals at each point for regression model  $y = 4x - 4$ .

x	y	$y_{\text{predicted}}$	$\varepsilon = y - y_{\text{predicted}}$
2.0	4.0	4.0	0.0
3.0	6.0	8.0	-2.0
2.0	6.0	4.0	2.0
3.0	8.0	8.0	0.0
			$\sum_{i=1}^4 \varepsilon_i = 0$



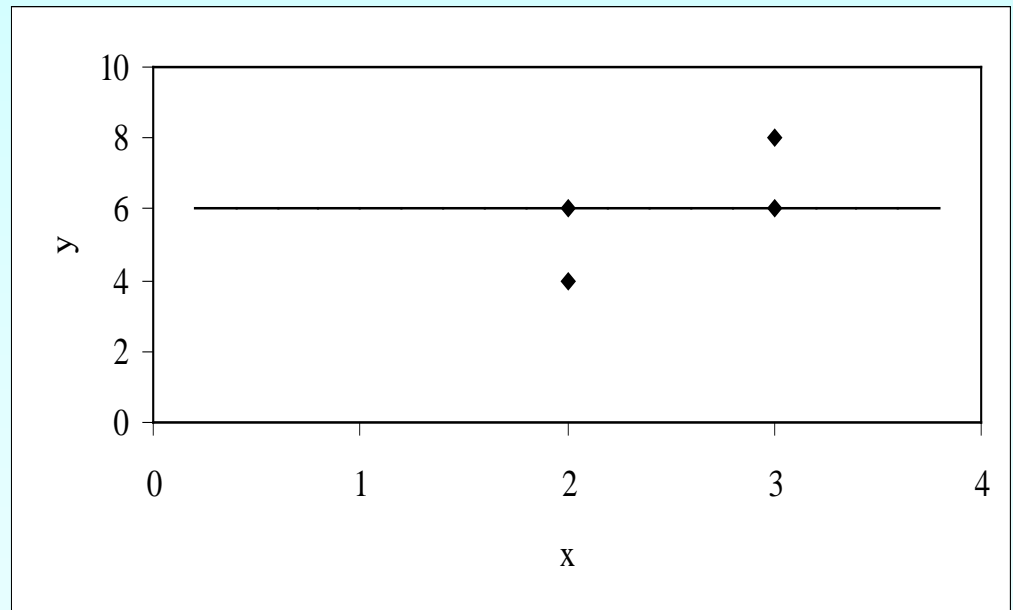
**Figure.** Regression curve for  $y=4x-4$ ,  $y$  vs.  $x$  data

# Linear Regression-Criteria#1

Using  $y=6$  as a regression curve

**Table.** Residuals at each point for  $y=6$

x	y	$y_{\text{predicted}}$	$\varepsilon = y - y_{\text{predicted}}$
2.0	4.0	6.0	-2.0
3.0	6.0	6.0	0.0
2.0	6.0	6.0	0.0
3.0	8.0	6.0	2.0
			$\sum_{i=1}^4 \varepsilon_i = 0$



**Figure.** Regression curve for  $y=6$ ,  $y$  vs.  $x$  data

# Linear Regression – Criterion #1

$$\sum_{i=1}^4 \varepsilon_i = 0 \quad \text{for both regression models of } y=4x-4 \text{ and } y=6.$$

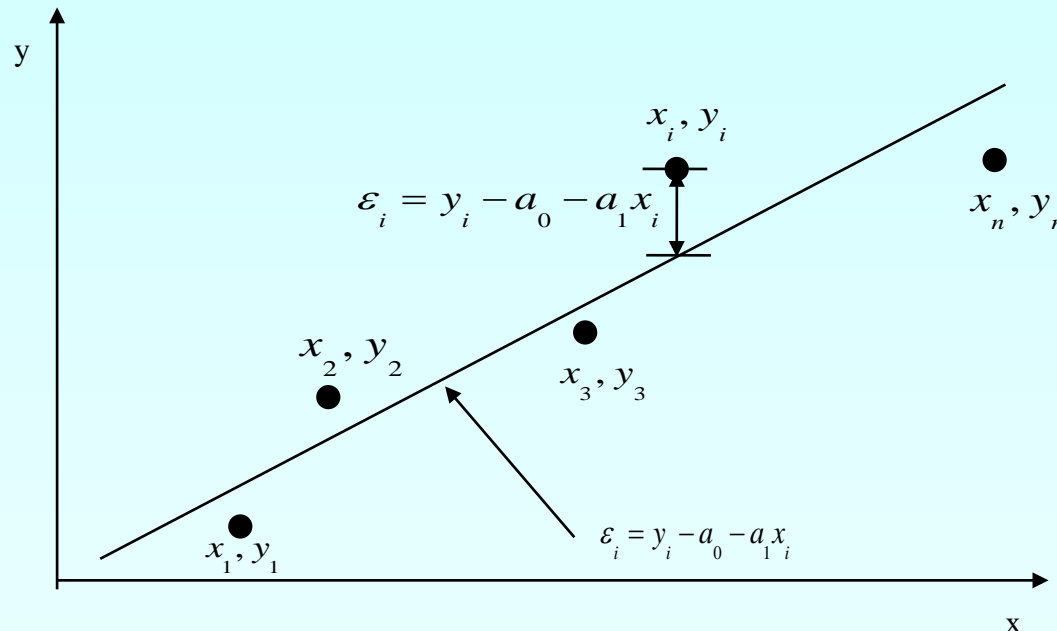
The sum of the residuals is as small as possible, that is zero, but the regression model is not unique.

Hence the above criterion of minimizing the sum of the residuals is a bad criterion.



# Linear Regression-Criterion#2

Will minimizing  $\sum_{i=1}^n |\varepsilon_i|$  work any better?



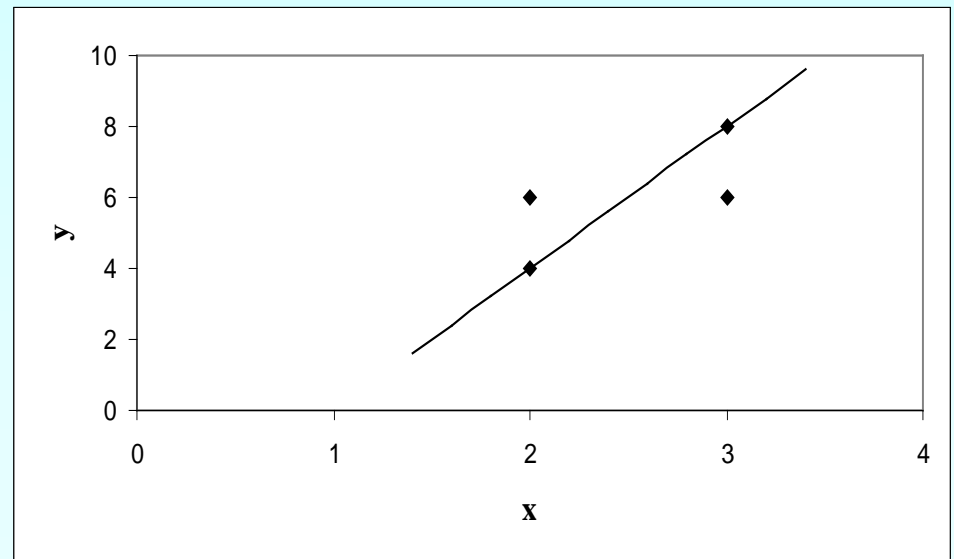
**Figure.** Linear regression of  $y$  vs.  $x$  data showing residuals at a typical point,  $x_i$ .

# Linear Regression-Criteria 2

Using  $y=4x-4$  as the regression curve

**Table.** The absolute residuals employing the  $y=4x-4$  regression model

x	y	$y_{\text{predicted}}$	$ \epsilon  =  y - y_{\text{predicted}} $
2.0	4.0	4.0	0.0
3.0	6.0	8.0	2.0
2.0	6.0	4.0	2.0
3.0	8.0	8.0	0.0
			$\sum_{i=1}^4  \epsilon_i  = 4$



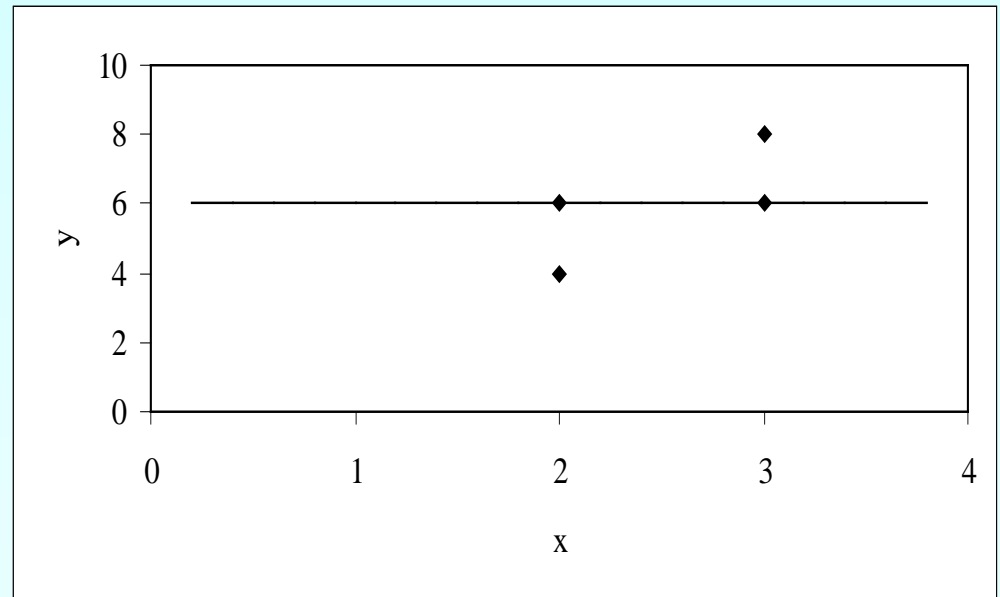
**Figure.** Regression curve for  $y=4x-4$ ,  $y$  vs.  $x$  data

# Linear Regression-Criteria#2

Using  $y=6$  as a regression curve

**Table.** Absolute residuals employing the  $y=6$  model

x	y	$y_{\text{predicted}}$	$ \varepsilon  =  y - y_{\text{predicted}} $
2.0	4.0	6.0	2.0
3.0	6.0	6.0	0.0
2.0	6.0	6.0	0.0
3.0	8.0	6.0	2.0
			$\sum_{i=1}^4  \varepsilon_i  = 4$



**Figure.** Regression curve for  $y=6$ ,  $y$  vs.  $x$  data

# Linear Regression-Criterion#2

$$\sum_{i=1}^4 |\varepsilon_i| = 4 \text{ for both regression models of } y=4x-4 \text{ and } y=6.$$

The sum of the errors has been made as small as possible, that is 4, but the regression model is not unique.

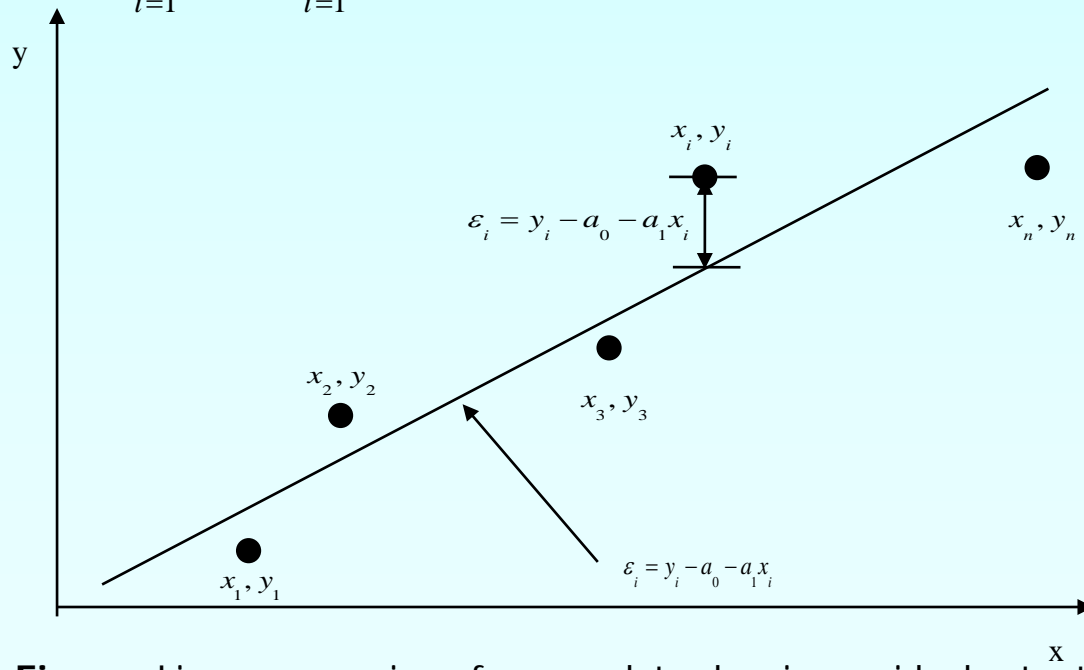
Hence the above criterion of minimizing the sum of the absolute value of the residuals is also a bad criterion.

Can you find a regression line for which  $\sum_{i=1}^4 |\varepsilon_i| < 4$  and has unique regression coefficients?

# Least Squares Criterion

The least squares criterion minimizes the sum of the square of the residuals in the model, and also produces a unique line.

$$S_r = \sum_{i=1}^n \varepsilon_i^2 = \sum_{i=1}^n (y_i - a_0 - a_1 x_i)^2$$



**Figure.** Linear regression of  $y$  vs.  $x$  data showing residuals at a typical point,  $x_i$ .

# Finding Constants of Linear Model

Minimize the sum of the square of the residuals:  $S_r = \sum_{i=1}^n \varepsilon_i^2 = \sum_{i=1}^n (y_i - a_0 - a_1 x_i)^2$

To find  $a_0$  and  $a_1$  we minimize  $S_r$  with respect to  $a_1$  and  $a_0$ .

$$\frac{\partial S_r}{\partial a_0} = -2 \sum_{i=1}^n (y_i - a_0 - a_1 x_i)(-1) = 0$$

$$\frac{\partial S_r}{\partial a_1} = -2 \sum_{i=1}^n (y_i - a_0 - a_1 x_i)(-x_i) = 0$$

giving

$$\sum_{i=1}^n a_0 + \sum_{i=1}^n a_1 x_i = \sum_{i=1}^n y_i$$

$$(a_0 = \bar{y} - a_1 \bar{x})$$

$$\sum_{i=1}^n a_0 x_i + \sum_{i=1}^n a_1 x_i^2 = \sum_{i=1}^n y_i x_i$$

# Finding Constants of Linear Model

Solving for  $a_0$  and  $a_1$  directly yields,

$$a_1 = \frac{n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2}$$

and

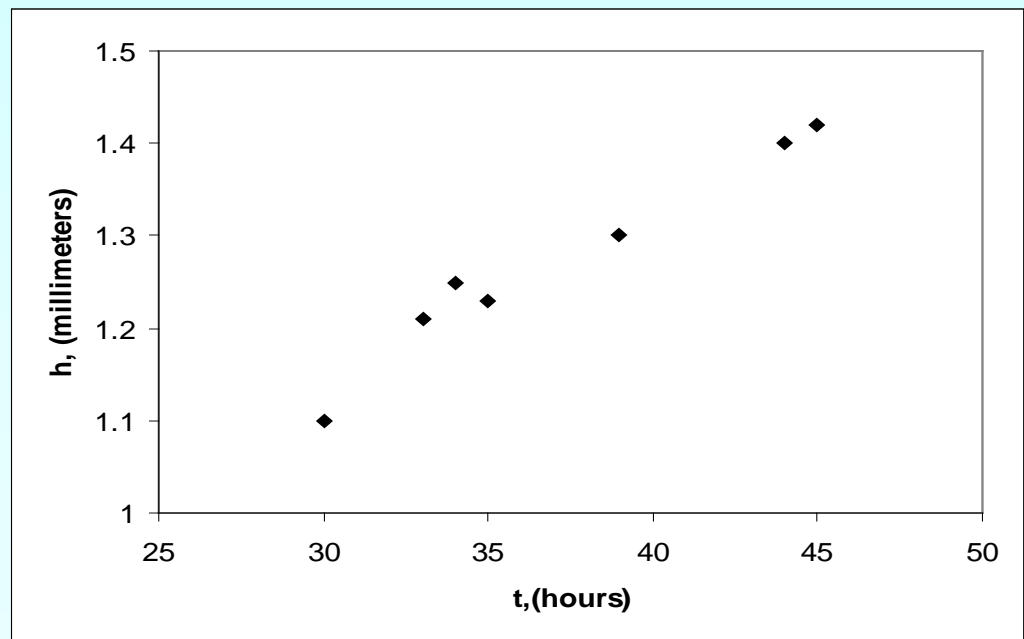
$$a_0 = \frac{\sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i - \sum_{i=1}^n x_i \sum_{i=1}^n x_i y_i}{n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2} \quad (a_0 = \bar{y} - a_1 \bar{x})$$

# Example 1

As machines are used over long periods of time, the output product can get off target. Below is the average value of how much off target a product is getting manufactured as a function of machine use.

**Table.** Off target value as a function of machine use.

Hours of Machine Use, $t$	Millimeters Off Target, $h$
30	1.10
33	1.21
34	1.25
35	1.23
39	1.30
44	1.40
45	1.42



**Figure.** Data points for  $h$  vs.  $t$  data



# Example 1 cont.

Regress the data to  $h = a_0 + a_1 t$  and find when the product will be 2mm off target.

**Table.** Summation data for linear regression

$t$	$h$	$t^2$	$t \times h$
<i>Hours</i>	<i>Millimeters</i>	<i>Hours<sup>2</sup></i>	<i>Millimeter-Hour</i>
30	1.10	900	33
33	1.21	1089	39.93
34	1.25	1156	42.50
35	1.23	1225	43.05
39	1.30	1521	50.70
44	1.40	1936	61.6
45	1.42	2025	63.9
260	8.91	9852	334.68

$$\sum_{i=1}^7$$

With  $n = 7$

$$\begin{aligned}
 a_1 &= \frac{n \sum_{i=1}^7 t_i h_i - \sum_{i=1}^7 t_i \sum_{i=1}^7 h_i}{n \sum_{i=1}^7 t_i^2 - \left( \sum_{i=1}^7 t_i \right)^2} \\
 &= \frac{7(334.68) - (260)(8.91)}{7(9852) - (260)^2} \\
 &= 0.019179 \text{ mm} - h
 \end{aligned}$$

# Example 1 cont.

The value for  $a_0$  can then be found using  $a_0 = \bar{h} - a_1 \bar{t}$  where

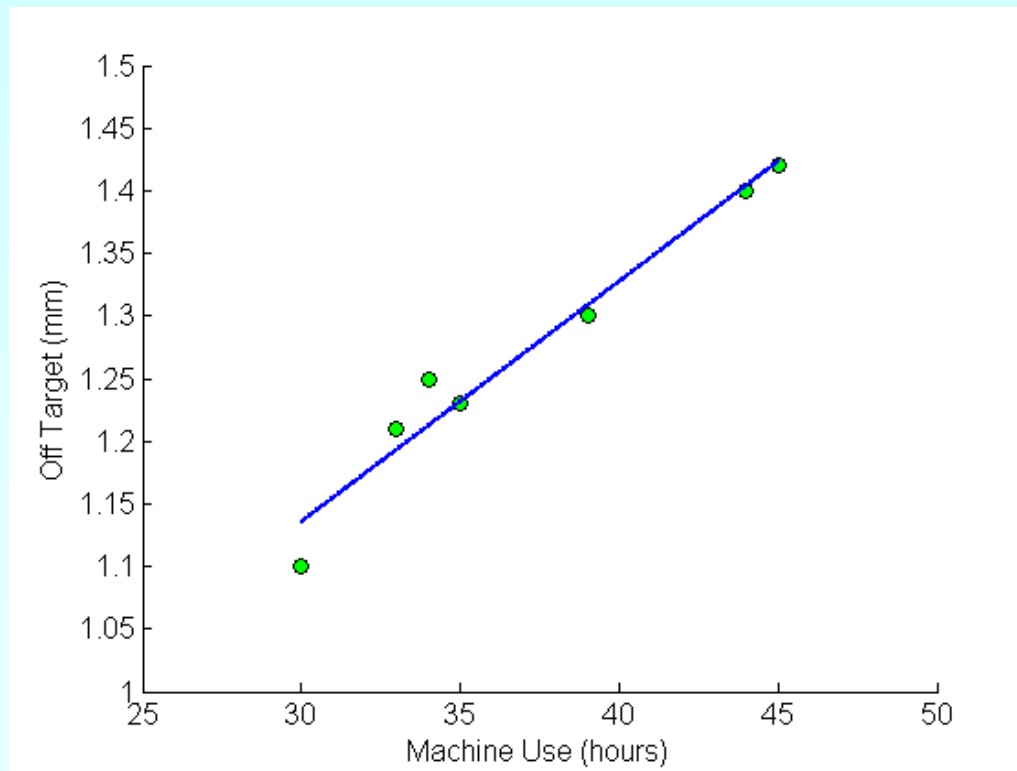
$$\bar{h} = \frac{\sum_{i=1}^7 h_i}{n} = 1.2729 \text{ mm} \qquad \bar{t} = \frac{\sum_{i=1}^7 t_i}{n} = 37.143 \text{ hours}$$

$$\begin{aligned} a_0 &= \bar{h} - a_1 \bar{t} \\ &= 1.2729 - (0.01979)(37.143) \\ &= 0.56050 \text{ mm} - \text{h} \end{aligned}$$

$$h = 0.56050 + 0.019179t$$

# Example 1 cont.

The linear regression model is now given by  $h = 0.56050 + 0.019179t$



**Figure.** Linear regression of hours of use vs. millimeters off target.

# Example 1 cont.

Solving for when  $h = 2$  mm yields

$$h = 0.56050 + 0.19179t$$

$$2 = 0.56050 + 0.019179t$$

$$\begin{aligned} t &= \frac{2 - 0.56050}{0.019179} \\ &= 75.056 \text{ hours} \end{aligned}$$

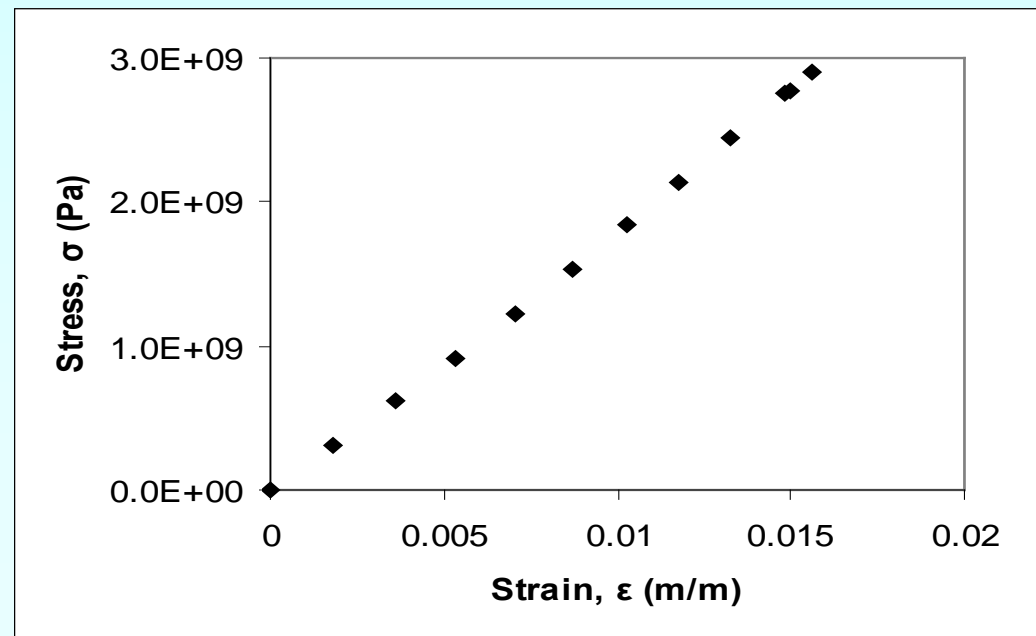
# Example 2

To find the longitudinal modulus of composite, the following data is collected. Find the longitudinal modulus,  $E$  using the regression model

**Table.** Stress vs. Strain data

Strain (%)	Stress (MPa)
0	0
0.183	306
0.36	612
0.5324	917
0.702	1223
0.867	1529
1.0244	1835
1.1774	2140
1.329	2446
1.479	2752
1.5	2767
1.56	2896

$\sigma = E\varepsilon$  and the sum of the square of the residuals.



**Figure.** Data points for Stress vs. Strain data

## Example 2 cont.

Residual at each point is given by

$$\gamma_i = \sigma_i - E\varepsilon_i$$

The sum of the square of the residuals then is

$$\begin{aligned} S_r &= \sum_{i=1}^n \gamma_i^2 \\ &= \sum_{i=1}^n (\sigma_i - E\varepsilon_i)^2 \end{aligned}$$

Differentiate with respect to  $E$

$$\frac{\partial S_r}{\partial E} = \sum_{i=1}^n 2(\sigma_i - E\varepsilon_i)(-\varepsilon_i) = 0$$

Therefore

$$E = \frac{\sum_{i=1}^n \sigma_i \varepsilon_i}{\sum_{i=1}^n \varepsilon_i^2}$$

# Example 2 cont.

**Table.** Summation data for regression model

i	$\epsilon$	$\sigma$	$\epsilon^2$	$\epsilon\sigma$
1	0.0000	0.0000	0.0000	0.0000
2	1.8300 $10^{-3}$	3.0600 $10^8$	3.3489 $10^{-6}$	5.5998 $10^5$
3	3.6000 $10^{-3}$	6.1200 $10^8$	1.2960 $10^{-5}$	2.2032 $10^6$
4	5.3240 $10^{-3}$	9.1700 $10^8$	2.8345 $10^{-5}$	4.8821 $10^6$
5	7.0200 $10^{-3}$	1.2230 $10^9$	4.9280 $10^{-5}$	8.5855 $10^6$
6	8.6700 $10^{-3}$	1.5290 $10^9$	7.5169 $10^{-5}$	1.3256 $10^7$
7	1.0244 $10^{-2}$	1.8350 $10^9$	1.0494 $10^{-4}$	1.8798 $10^7$
8	1.1774 $10^{-2}$	2.1400 $10^9$	1.3863 $10^{-4}$	2.5196 $10^7$
9	1.3290 $10^{-2}$	2.4460 $10^9$	1.7662 $10^{-4}$	3.2507 $10^7$
10	1.4790 $10^{-2}$	2.7520 $10^9$	2.1874 $10^{-4}$	4.0702 $10^7$
11	1.5000 $10^{-2}$	2.7670 $10^9$	2.2500 $10^{-4}$	4.1505 $10^7$
12	1.5600 $10^{-2}$	2.8960 $10^9$	2.4336 $10^{-4}$	4.5178 $10^7$
$\sum_{i=1}^{12}$			1.2764 $10^{-3}$	2.3337 $10^8$

With

$$\sum_{i=1}^{12} \epsilon_i^2 = 1.2764 \times 10^{-3}$$

and

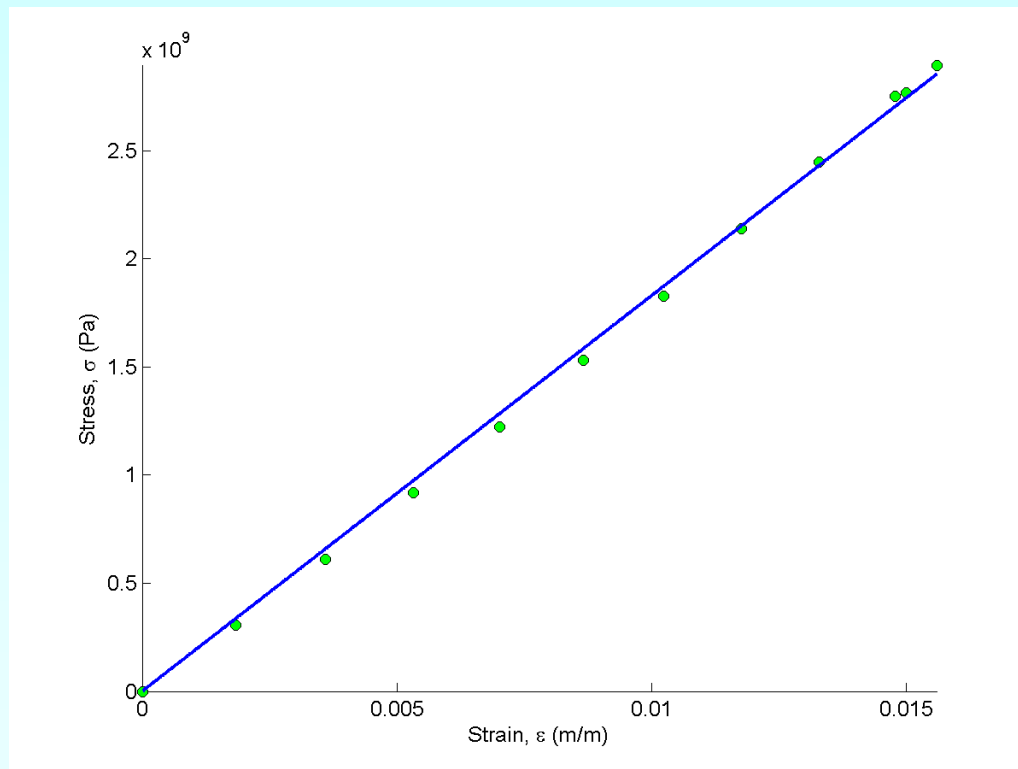
$$\sum_{i=1}^{12} \sigma_i \epsilon_i = 2.3337 \times 10^8$$

Using

$$\begin{aligned}
 E &= \frac{\sum_{i=1}^{12} \sigma_i \epsilon_i}{\sum_{i=1}^{12} \epsilon_i^2} \\
 &= \frac{2.3337 \times 10^8}{1.2764 \times 10^{-3}} \\
 &= 182.84 \text{ GPa}
 \end{aligned}$$

# Example 2 Results

The equation  $\sigma = 182.84\varepsilon$  describes the data.



**Figure.** Linear regression for Stress vs. Strain data



# Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

[http://numericalmethods.eng.usf.edu/topics/linear\\_regression.html](http://numericalmethods.eng.usf.edu/topics/linear_regression.html)

**THE END**

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