Spline Interpolation Method

Mechanical Engineering Majors

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Spline Method of Interpolation

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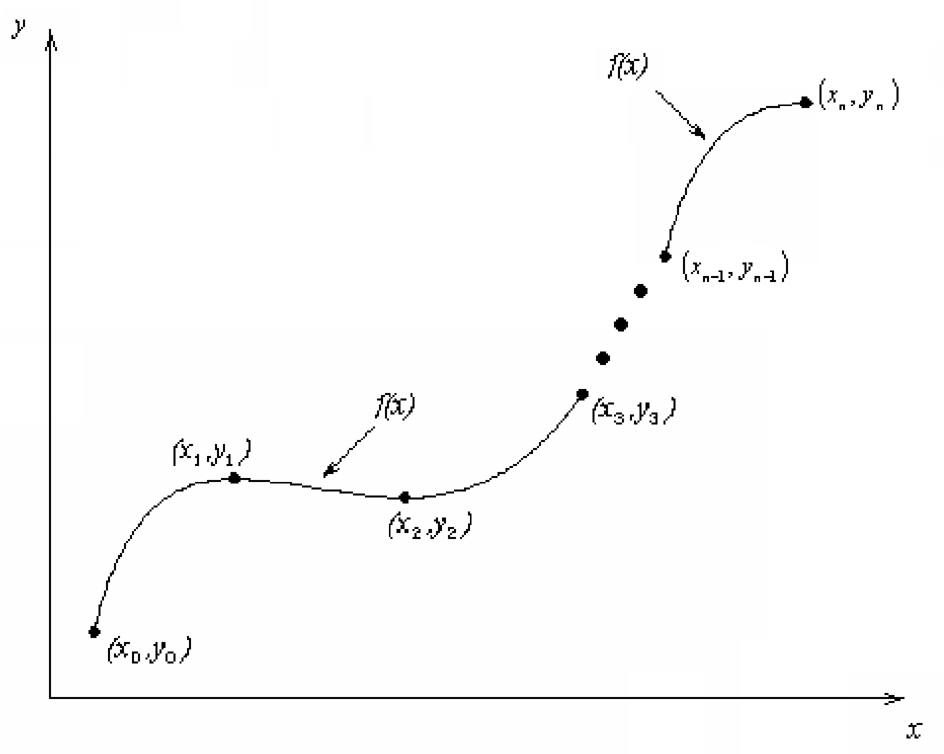
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Why Splines ?

$$f(x) = \frac{1}{1+25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

x	$y = \frac{1}{1+25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

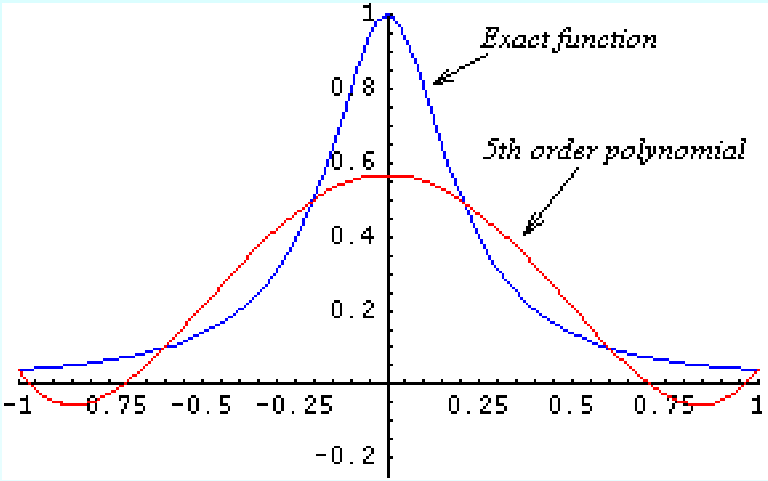


Figure : 5th order polynomial vs. exact function

Why Splines ?

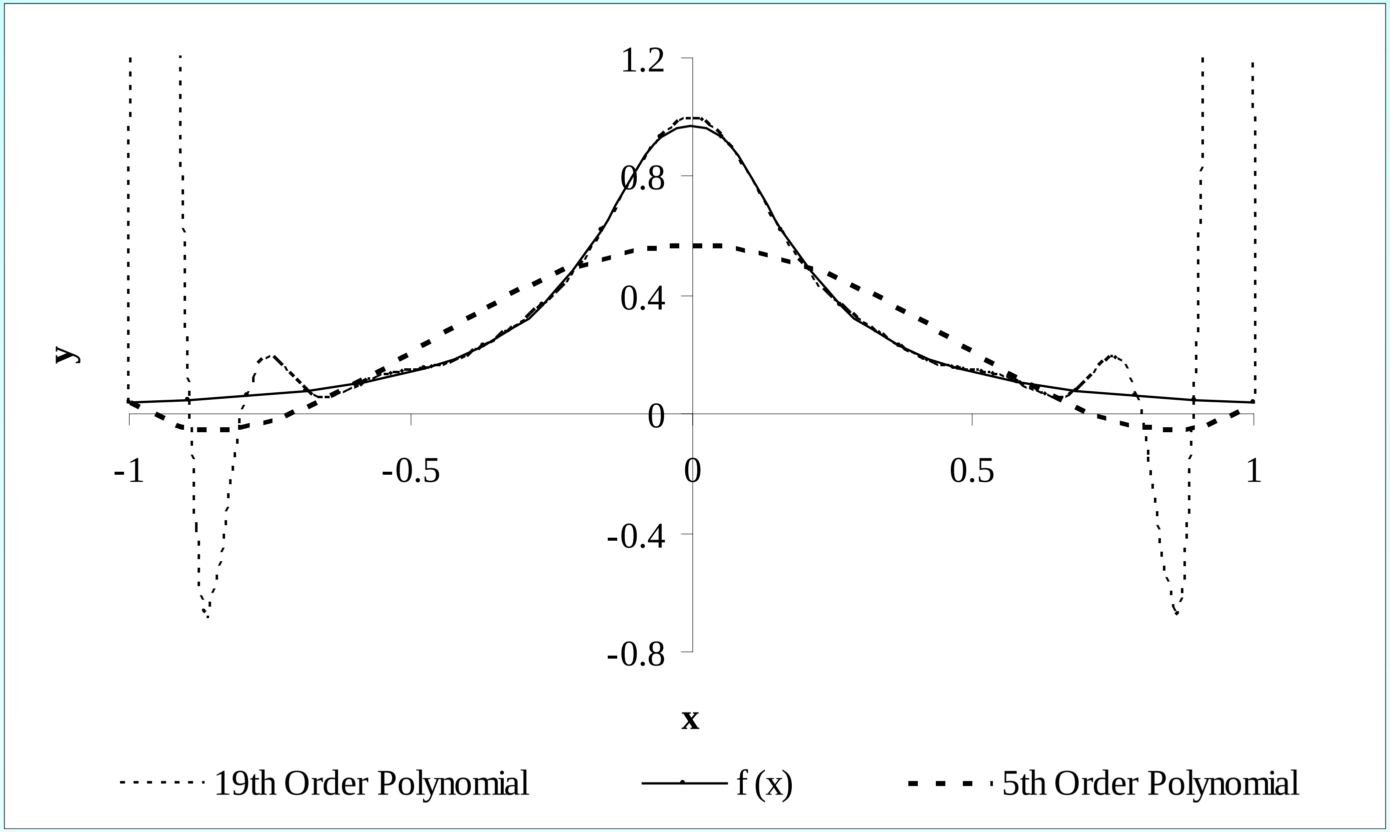
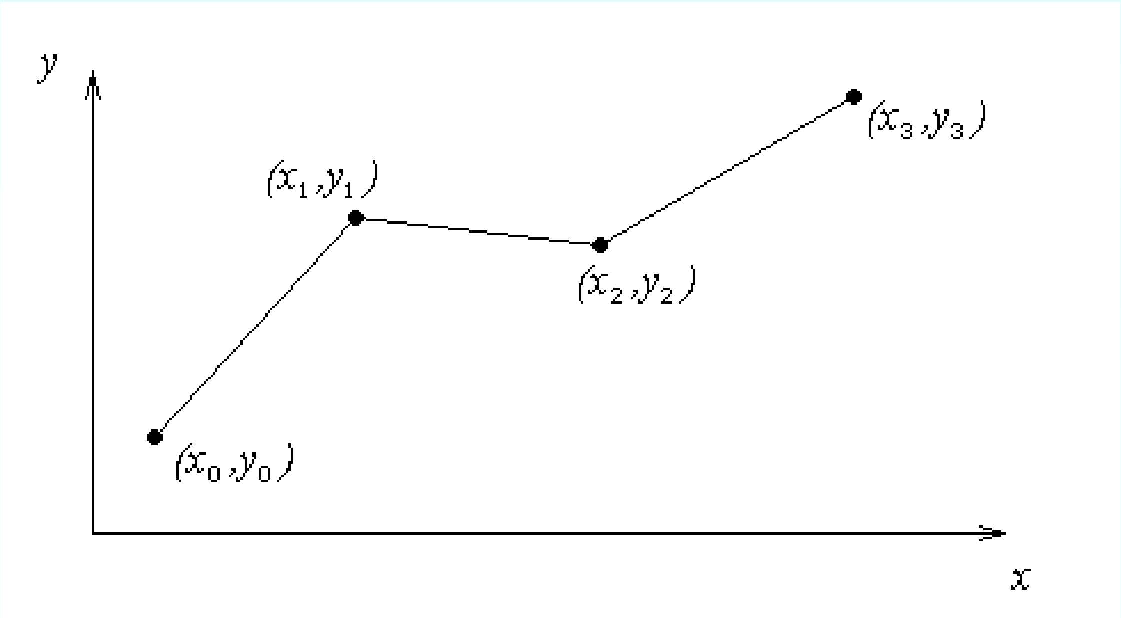


Figure : Higher order polynomial interpolation is a bad idea

Linear Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1})(x_n, y_n)$, fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $(y_i = f(x_i))$

Figure : Linear splines



Linear Interpolation (contd)

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2$$

.

.

.

$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n$$

Note the terms of

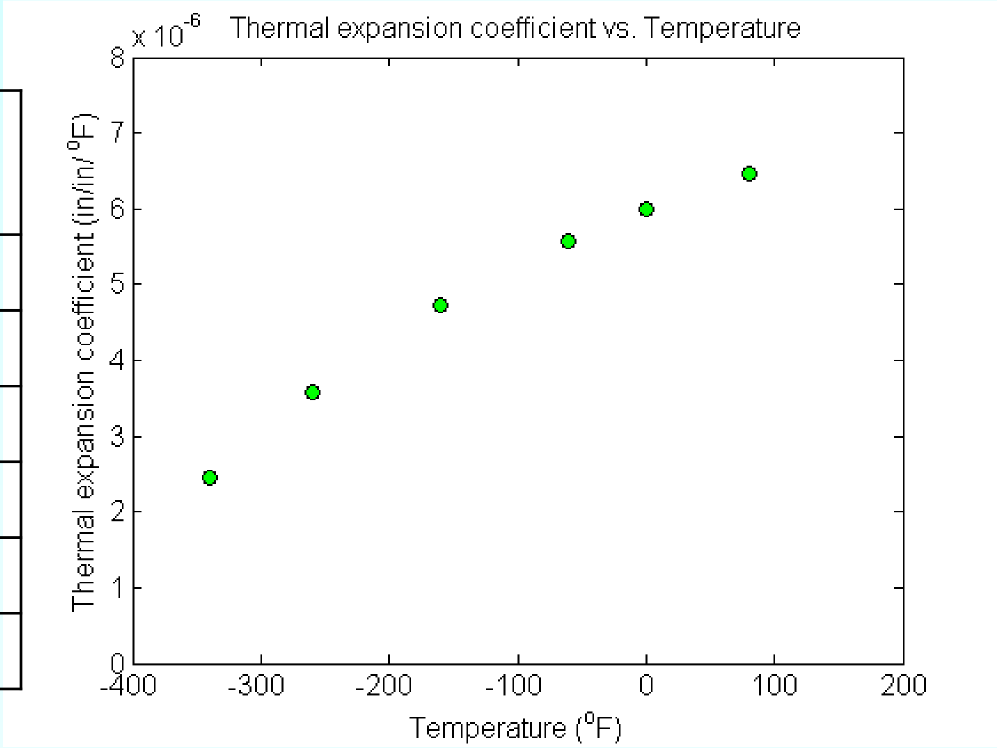
$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between x_{i-1} and x_i.

Example

A trunnion is cooled 80°F to − 108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Determine the value of the coefficient of thermal expansion at T=−14°F using linear spline interpolation.

Temperature (°F)	Thermal Expansion Coefficient (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
−60	5.58×10^{-6}
−160	4.72×10^{-6}
−260	3.58×10^{-6}
−340	2.45×10^{-6}



Linear Interpolation

$$T_0 = 0, \qquad \alpha(T_0) = 6.00\times10^{-6}$$

$$T_1 = -60, \qquad \alpha(T_1) = 5.58\times10^{-6}$$

$$\alpha(T) = \alpha(T_0) + \frac{\alpha(T_1) - \alpha(T_0)}{T_1 - T_0}(T - T_0)$$

$$= 6.00\times10^{-6} + \frac{5.58\times10^{-6} - 6.00\times10^{-6}}{-60-0}(T-0)$$

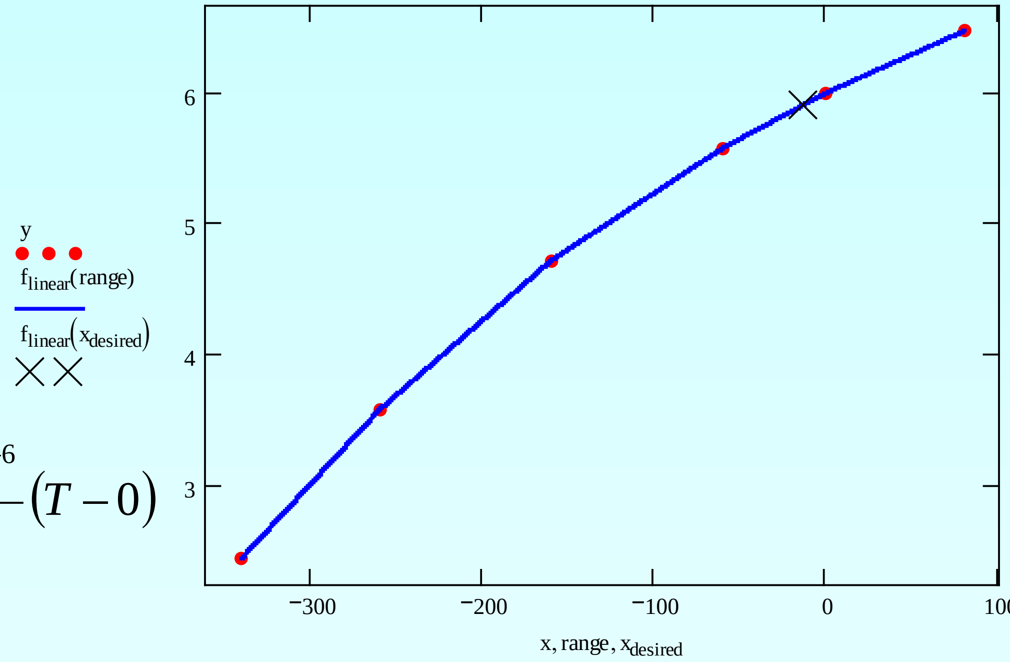
$$\alpha(T) = 6.00\times10^{-6} + 0.007\times10^{-6}(T-0)$$

$$-60 \le T \le 0$$

At $T = -14$,

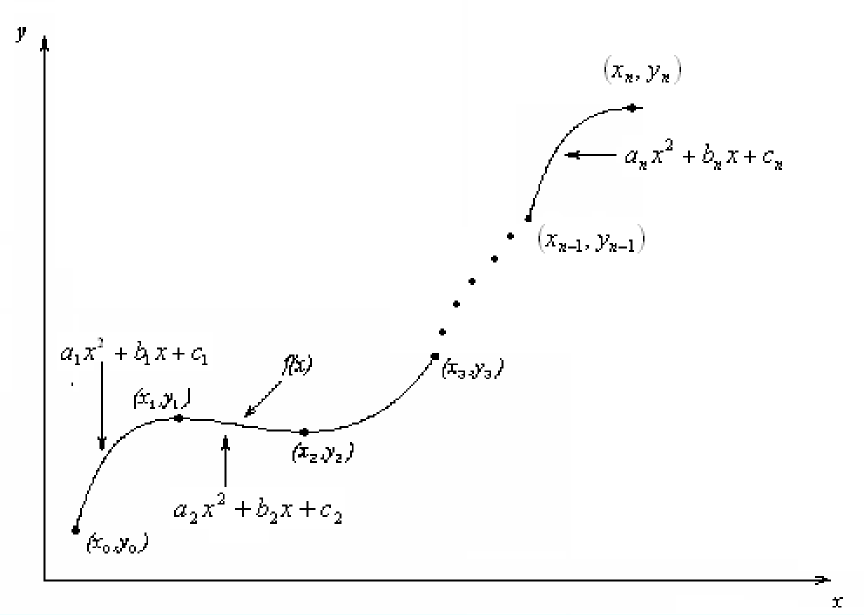
$$\alpha(-14) = 6.00\times10^{-6} + 0.007\times10^{-6}(-14-0)$$

$$= 5.902\times10^{-6} \text{ in/in/°F}$$



Quadratic Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by



$$
\begin{aligned}
f(x) &= a_1 x^2 + b_1 x + c_1, & x_0 \le x \le x_1 \\
&= a_2 x^2 + b_2 x + c_2, & x_1 \le x \le x_2 \\
&. \\
&. \\
&. \\
&= a_n x^2 + b_n x + c_n, & x_{n-1} \le x \le x_n
\end{aligned}
$$

Find $a_i, b_i, c_i, i = 1, 2, \ldots, \mathrm{n}$

Quadratic Interpolation (contd)

Each quadratic spline goes through two consecutive data points

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$

$$a_1 x_1^2 + b_1 x_1 + c_1 = f(x_1)$$

.

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$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

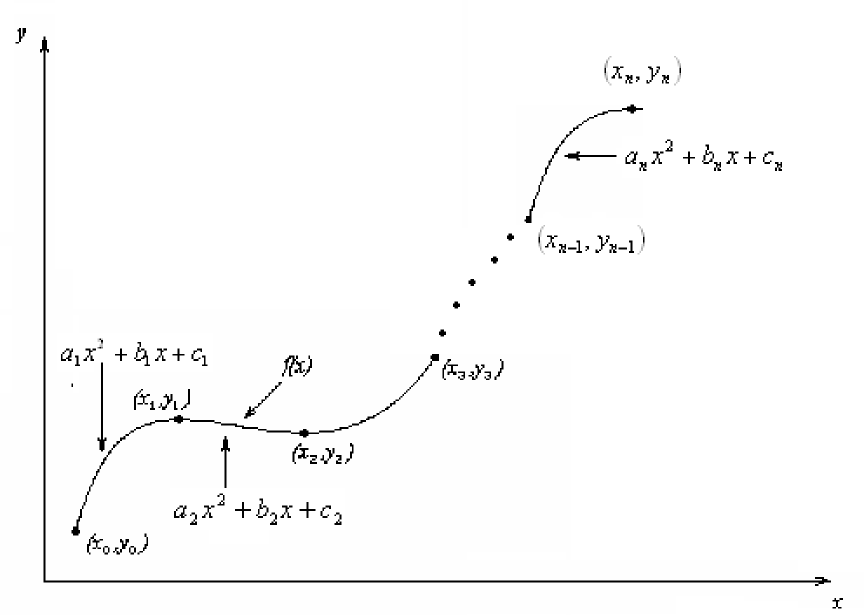
$$a_i x_i^2 + b_i x_i + c_i = f(x_i)$$

.

.

$$a_n x_{n-1}^2 + b_n x_{n-1} + c_n = f(x_{n-1})$$

$$a_n x_n^2 + b_n x_n + c_n = f(x_n)$$



This condition gives 2n equations

Quadratic Splines (contd)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1 \quad \text{is} \quad 2a_1x + b_1$$

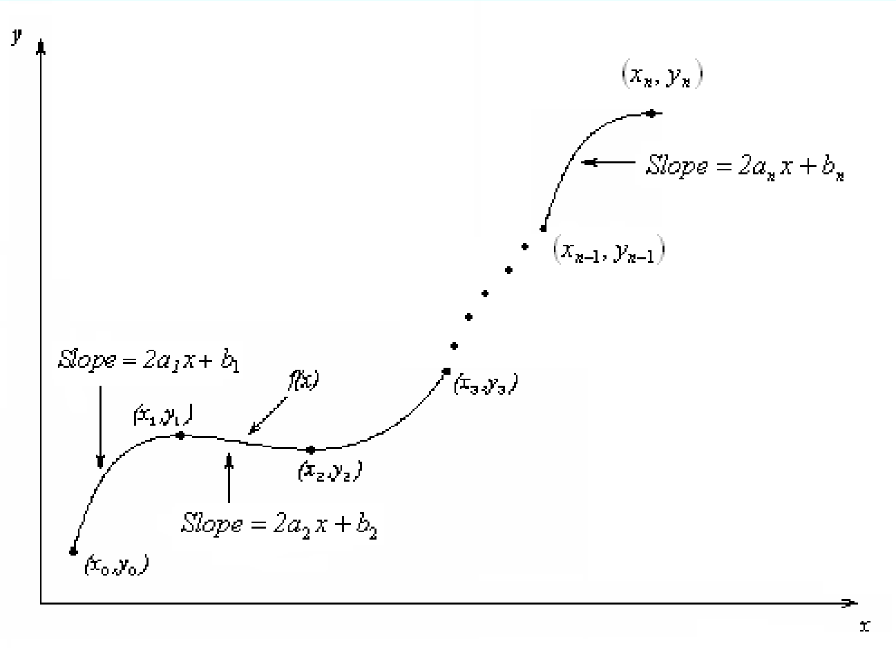
The derivative of the second spline

$$a_2x^2 + b_2x + c_2 \quad \text{is} \quad 2a_2x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



Quadratic Splines (contd)

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

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$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

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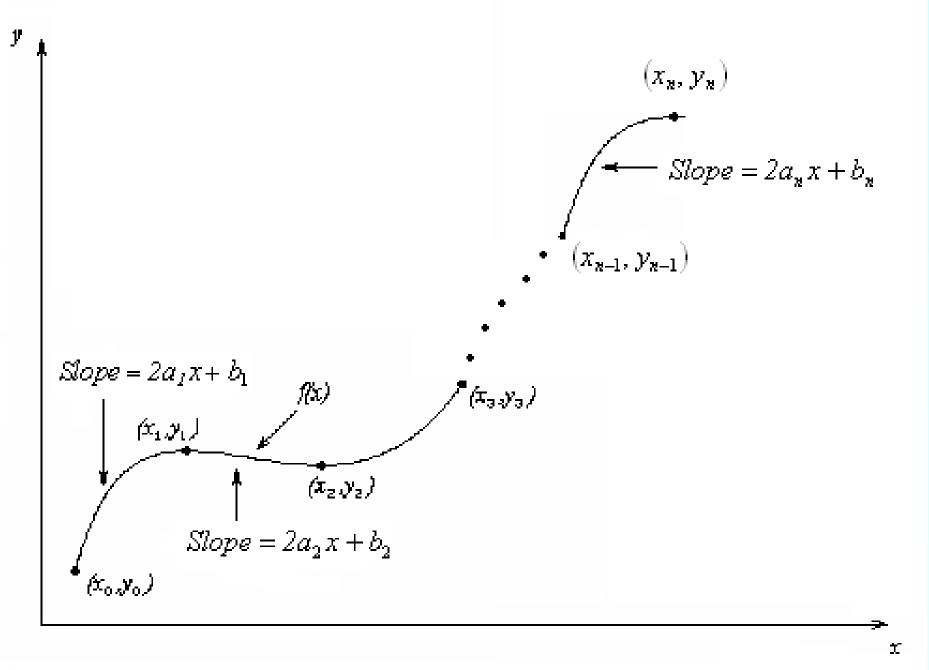
$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$



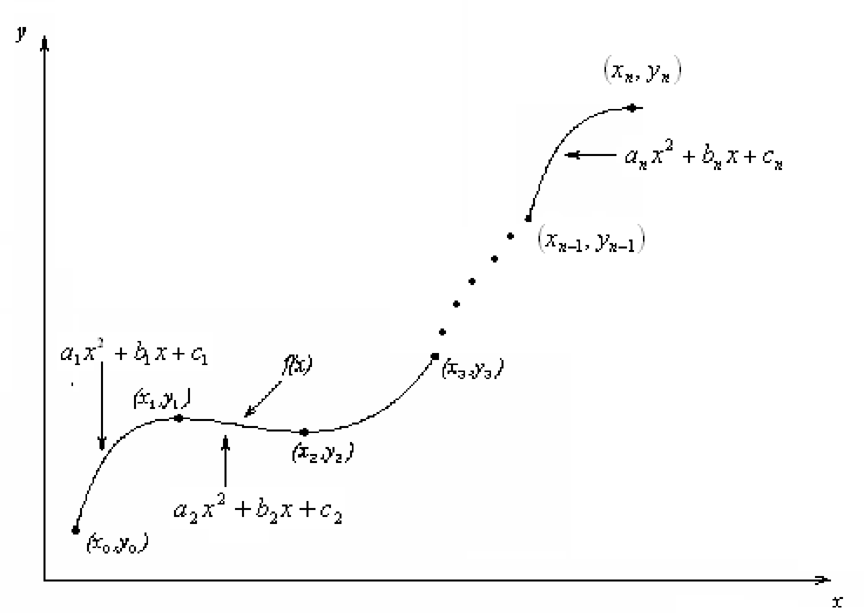
We have (n-1) such equations. The total number of equations is $(2n) + (n-1) = (3n-1)$.

We can assume that the first spline is linear, that is $a_1 = 0$

Quadratic Splines (contd)

This gives us ‘3n’ equations and ‘3n’ unknowns. Once we find the ‘3n’ constants, we can find the function at any value of ‘x’ using the splines,

$$
\begin{aligned}
f(x) &= a_1x^2 + b_1x + c_1, & x_0 \le x \le x_1 \\
&= a_2x^2 + b_2x + c_2, & x_1 \le x \le x_2 \\
&. \\
&. \\
&. \\
&= a_nx^2 + b_nx + c_n, & x_{n-1} \le x \le x_n
\end{aligned}
$$



Example

A trunnion is cooled 80°F to − 108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Determine the value of the coefficient of thermal expansion at T=−14°F using quadratic spline interpolation.

Temperature (°F)	Thermal Expansion Coefficient (in/in/°F)
80	6.47×10^{-6}
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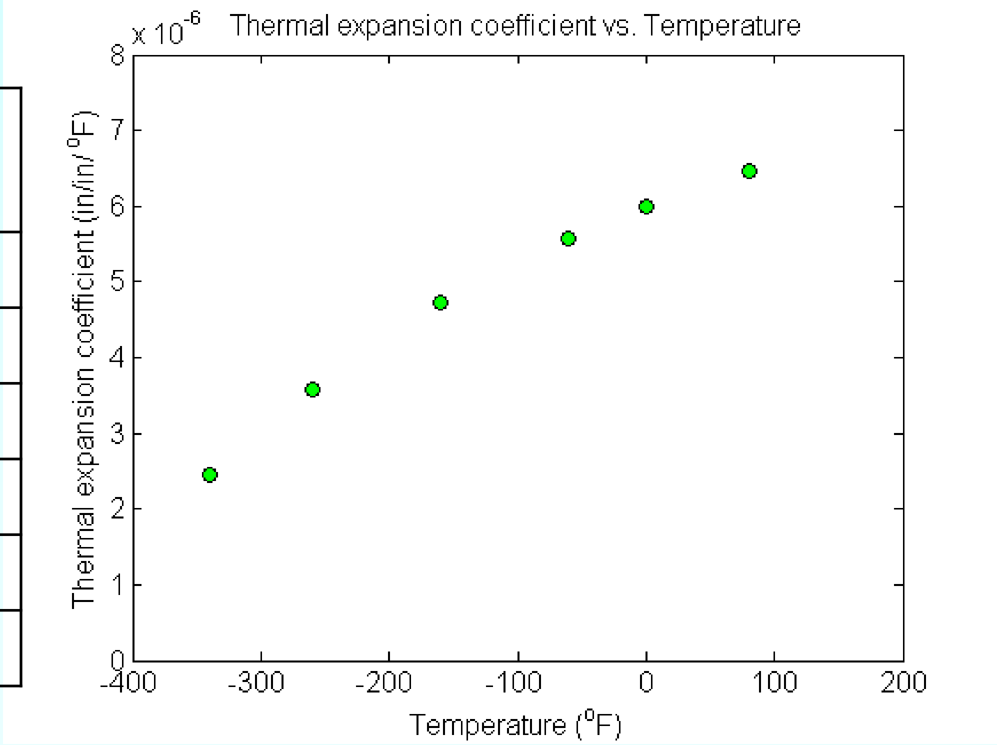


Solution

Since there are six data points,

five quadratic splines pass through them.

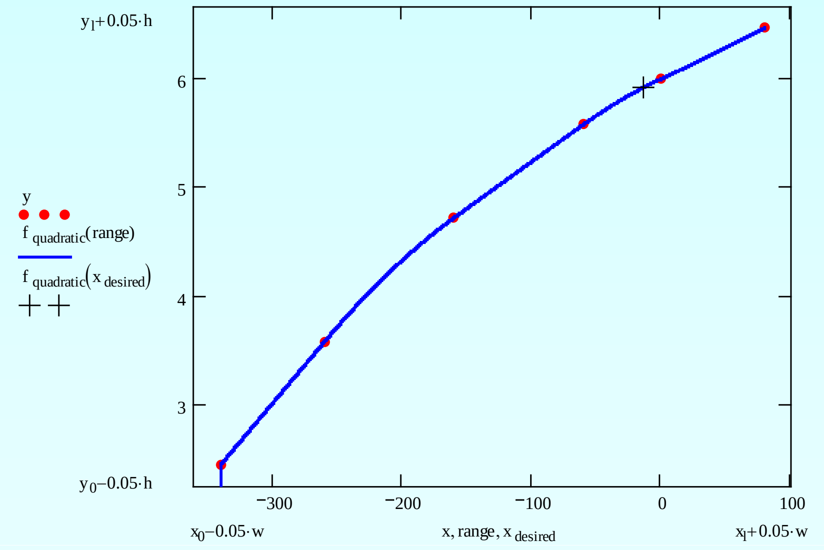
$$\alpha(T) = a_1T^2 + b_1T + c_1, \qquad -340 \le T \le -260$$

$$= a_2T^2 + b_2T + c_2, \qquad -260 \le T \le -160$$

$$= a_3T^2 + b_3T + c_3, \qquad -160 \le T \le -60$$

$$= a_4T^2 + b_4T + c_4, \qquad -60 \le T \le 0$$

$$= a_5T^2 + b_5T + c_5, \qquad 0 \le T \le 80$$



Solution (contd)

Setting up the equations

Each quadratic spline passes through two consecutive data points giving

$a_1T^2 + b_1T + c_1$ passes through T = −340 and T =−260,

$$a_1(-340)^2 + b_1(-340) + c_1 = 2.45\times10^{-6} \qquad (1)$$

$$a_1(-260)^2 + b_1(-260) + c_1 = 3.58\times10^{-6} \qquad (2)$$

Similarly,

$$a_2(-260)^2 + b_2(-260) + c_2 = 3.58\times10^{-6} \qquad (3)$$

$$a_2(-160)^2 + b_2(-160) + c_2 = 4.72\times10^{-6} \qquad (4)$$

$$a_3(-160)^2 + b_3(-160) + c_3 = 4.72\times10^{-6} \qquad (5)$$

$$a_3(-60)^2 + b_3(-60) + c_3 = 5.58\times10^{-6} \qquad (6)$$

$$a_4(-60)^2 + b_4(-60) + c_4 = 5.58\times10^{-6} \qquad (7)$$

$$a_4(0)^2 + b_4(0) + c_4 = 6.00\times10^{-6} \qquad (8)$$

$$a_5(0)^2 + b_5(0) + c_5 = 6.00\times10^{-6} \qquad (9)$$

$$a_5(80)^2 + b_5(80) + c_5 = 6.47\times10^{-6} \qquad (10)$$

Solution (contd)

Quadratic splines have continuous derivatives at the interior data points

At $T = -260$

$$2a_1(-260) + b_1 - 2a_2(-260) - b_2 = 0 \quad (11)$$

At $T = -160$

$$2a_2(-160) + b_2 - 2a_3(-160) - b_3 = 0 \quad (12)$$

At $T = -60$

$$2a_3(-60) + b_3 - 2a_4(-60) - b_4 = 0 \quad (13)$$

At $T = 0$

$$2a_4(0) + b_4 - 2a_5(0) - b_5 = 0 \quad (14)$$

Assuming the first spline $a_1T^2 + b_1T + c_1$ is linear,

$$a_1 = 0 \quad (15)$$

Solution (contd)

$$
\begin{bmatrix}
1.156\times10^{5} & -340 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
67600 & -260 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 67600 & -260 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 25600 & -160 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 25600 & -160 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3600 & -60 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3600 & -60 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6400 & 80 & 1 \\
-520 & 1 & 0 & 520 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -320 & 1 & 0 & 320 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -120 & 1 & 0 & 120 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \end{bmatrix}
=
\begin{bmatrix} 2.45\times10^{-6} \\ 3.58\times10^{-6} \\ 3.58\times10^{-6} \\ 4.72\times10^{-6} \\ 4.72\times10^{-6} \\ 5.58\times10^{-6} \\ 5.58\times10^{-6} \\ 6.00\times10^{-6} \\ 6.00\times10^{-6} \\ 6.47\times10^{-6} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Solution (contd)

Solving the above 15 equations gives the 15 unknowns as

i	a_i	b_i	c_i
1	0	0.014125×10^{-6}	7.2525×10^{-6}
2	-2.725×10^{-11}	-4.5×10^{-11}	5.4104×10^{-6}
3	-7.5×10^{-13}	0.008435×10^{-6}	6.0888×10^{-6}
4	-2.5417×10^{-11}	0.005475×10^{-6}	6×10^{-6}
5	5×10^{-12}	0.005475×10^{-6}	6×10^{-6}

Solution (contd)

Therefore, the splines are given by

$$\alpha(T) = 0.014125\times10^{-6}T + 7.2525\times10^{-6}, \qquad -340 \le T \le -260$$

$$= -2.725\times10^{-11}T^2 - 4.5\times10^{-11}T + 5.4104\times10^{-6}, \qquad -260 \le T \le -160$$

$$= -7.5\times10^{-13}T^2 + 0.008435\times10^{-6}T + 6.0888\times10^{-6}, \qquad -160 \le T \le -60$$

$$= -2.5417\times10^{-11}T^2 + 0.005475\times10^{-6}T + 6\times10^{-6}, \qquad -60 \le T \le 0$$

$$= 5\times10^{-12}T^2 + 0.005475\times10^{-6}T + 6\times10^{-6}, \qquad 0 \le T \le 80$$

At T = −14

$$\alpha(-14) = -2.5417\times10^{-11}(-14)^2 + 0.005475\times10^{-6}(-14) + 6\times10^{-6}$$

$$= 5.9184\times10^{-6} \text{ in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the first order and second order polynomial is

$$|\in_a| = \left|\frac{5.9184\times10^{-6} - 5.902\times10^{-6}}{5.9184\times10^{-6}}\right|\times100$$

$$= 0.27657\%$$

Reduction in Diameter

The actual reduction in diameter is given by

$$\Delta D = D \int_{Tr}^{Tf} \alpha \, dT$$

where T_r = room temperature (°F)

T_f = temperature of cooling medium (°F)

Since T_r = 80 °F and T_r = −108 °F, $\Delta D = D \int_{80}^{-108} \alpha \, dT$

Find out the percentage difference in the reduction in the diameter by the above integral formula and the result using the thermal expansion coefficient from the cubic interpolation.

Reduction in Diameter

$$\int_{T_r}^{T_f} \alpha dT = \int_{80}^{-108} \alpha(T)dT = \int_{-60}^{-108} \alpha(T)dT + \int_{0}^{-60} \alpha(T)dT + \int_{80}^{0} \alpha(T)dT$$

$$= \int_{-60}^{-108} \left(-7.5\times10^{-13}T^2 + 0.008435\times10^{-6}T + 6.0888\times10^{-6}\right)dT$$

$$+ \int_{0}^{-60} \left(-2.5417\times10^{-11}T^2 + 0.005475\times10^{-6}T + 6\times10^{-6}\right)dT$$

$$+ \int_{80}^{0} \left(5\times10^{-12}T^2 + 0.005475\times10^{-6}T + 6\times10^{-6}\right)dT$$

$$= \left[-7.5\times10^{-13}\frac{T^3}{3} + 0.008435\times10^{-6}\frac{T^2}{2} + 6.0888\times10^{-6}T\right]_{-60}^{-108}$$

$$+ \left[-2.5417\times10^{-11}\frac{T^3}{3} + 0.005475\times10^{-6}\frac{T^2}{2} + 6\times10^{-6}T\right]_{0}^{-60}$$

$$+ \left[5\times10^{-12}\frac{T^3}{3} + 0.005475\times10^{-6}\frac{T^2}{2} + 6\times10^{-6}T\right]_{80}^{0}$$

Reduction in diameter

$$= [-257.99] \times 10^{-6} + [-348.32] \times 10^{-6} + [-498.373] \times 10^{-6}$$

$$\int_{T_r}^{T_f} \alpha dT = -1104.7 \times 10^{-6} \text{ in/in}$$

Taking the average coefficient of thermal expansion over this interval, given by:

$$\alpha_{avg} = \frac{\int_{T_r}^{T_f} \alpha dT}{T_f - T_r} = \frac{-1104.7 \times 10^{-6}}{-108 - 80} = 5.8760 \times 10^{-6} \text{ in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the 2nd methods is

$$|\in_a| = \left| \frac{5.8760 \times 10^{-6} - 5.9184 \times 10^{-6}}{5.8760 \times 10^{-6}} \right| \times 100$$

$$= 0.72178\%$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/spline_method.html

THE END