Chapter 05.02
Direct Method of Interpolation

After reading this chapter, you should be able to:

1. *apply the direct method of interpolation,*
2. *solve problems using the direct method of interpolation, and*
3. *use the direct method interpolants to find derivatives and integrals of discrete functions.*

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

(A) evaluate,
(B) differentiate, and
(C) integrate

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ points. One of the methods of interpolation is called the direct method. Other methods include Newton's divided difference polynomial method and the Lagrangian interpolation method. We will discuss the direct method in this chapter.

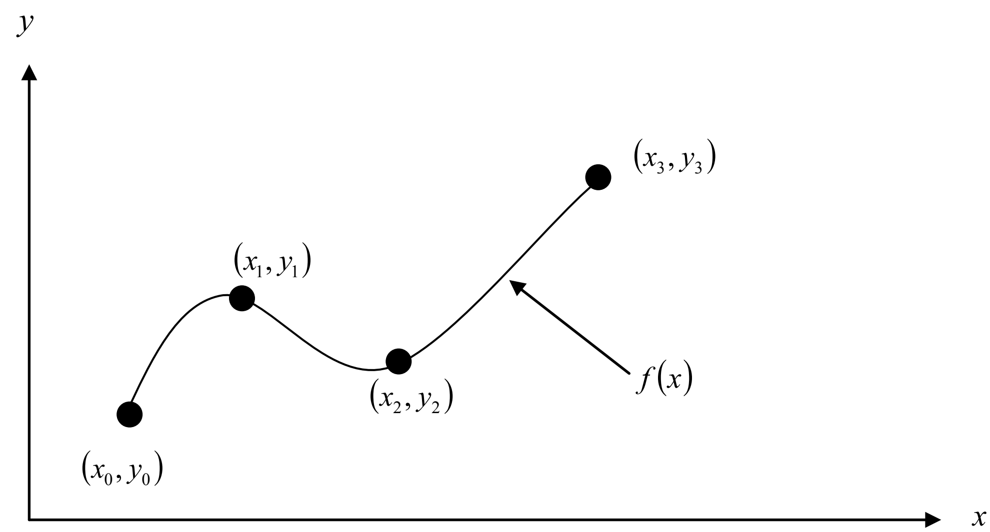

Figure 1 Interpolation of discrete data.

Direct Method

The direct method of interpolation is based on the following premise. Given $n+1$ data points, fit a polynomial of order n as given below

$$y = a_0 + a_1 x + ... + a_n x^n \tag{1}$$

through the data, where $a_0, a_1, ..., a_n$ are $n+1$ real constants. Since $n+1$ values of y are given at $n+1$ values of x, one can write $n+1$ equations. Then the $n+1$ constants, $a_0, a_1, ..., a_n$ can be found by solving the $n+1$ simultaneous linear equations. To find the value of y at a given value of x, simply substitute the value of x in Equation 1.

But, it is not necessary to use all the data points. How does one then choose the order of the polynomial and what data points to use? This concept and the direct method of interpolation are best illustrated using examples.

Example 1

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D \alpha \Delta T$$

where

$D =$ original diameter (in.)

$\alpha =$ coefficient of thermal expansion at average temperature (in/in/°F)

The trunnion is cooled from $80°\text{F}$ to $-108°\text{F}$, giving the average temperature as $-14°\text{F}$. The table of the coefficient of thermal expansion vs. temperature data is given in Table 1.

Table 1 Thermal expansion coefficient as a function of temperature.

Temperature, T $(°F)$	Thermal Expansion Coefficient, α (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
–60	5.58×10^{-6}
–160	4.72×10^{-6}
–260	3.58×10^{-6}
–340	2.45×10^{-6}

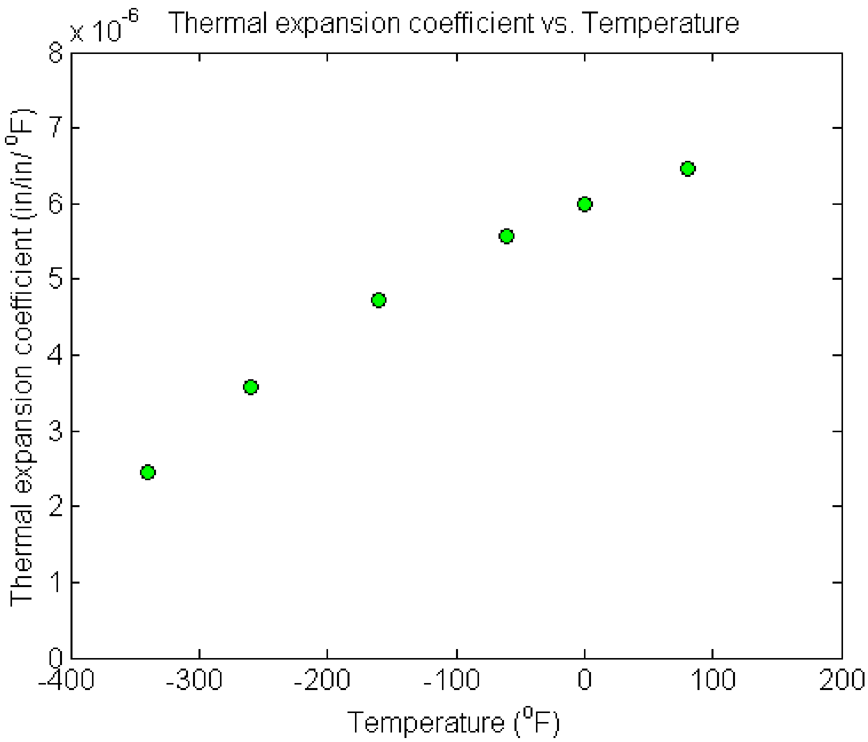

Figure 2 Thermal expansion coefficient vs. temperature.

If the coefficient of thermal expansion needs to be calculated at the average temperature of $-14°\text{F}$, determine the value of the coefficient of thermal expansion at $T=-14°\text{F}$ using the direct method of interpolation and a first order polynomial.

Solution

For first order polynomial interpolation (also called linear interpolation), we choose the coefficient of thermal expansion given by

$$\alpha(T)=a_0+a_1T$$

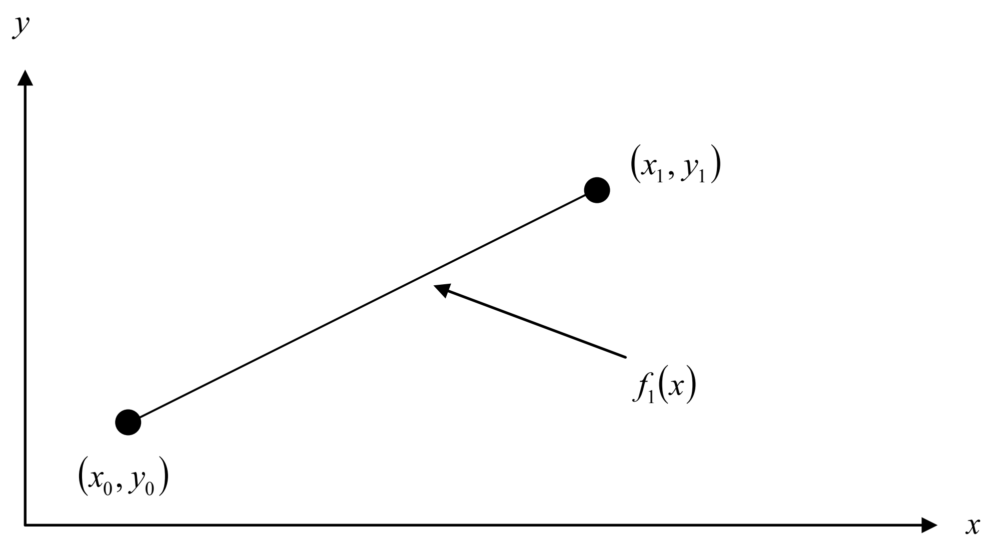

Figure 3 Linear interpolation.

Since we want to find the coefficient of thermal expansion at $T = -14\,°\text{F}$, and we are using a first order polynomial, we need to choose the two data points that are closest to $T = -14\,°\text{F}$ that also bracket $T = -14\,°\text{F}$ to evaluate it. The two points are $T_0 = 0\,°\text{F}$ and $T_1 = -60\,°\text{F}$.

Then

$$T_0 = 0, \quad \alpha(T_0) = 6.00 \times 10^{-6}$$
$$T_1 = -60, \; \alpha(T_1) = 5.58 \times 10^{-6}$$

gives

$$\alpha(0) = a_0 + a_1(0) = 6.00 \times 10^{-6}$$
$$\alpha(-60) = a_0 + a_1(-60) = 5.58 \times 10^{-6}$$

Writing the equations in matrix form, we have

$$\begin{bmatrix} 1 & 0 \\ 1 & -60 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 6.00 \times 10^{-6} \\ 5.58 \times 10^{-6} \end{bmatrix}$$

Solving the above two equations gives

$$a_0 = 6.00 \times 10^{-6}$$
$$a_1 = 0.007 \times 10^{-6}$$

Hence

$$\alpha(T) = a_0 + a_1 T$$
$$= 6.00 \times 10^{-6} + 0.007 \times 10^{-6} T, \; -60 \le T \le 0$$

At $T = -14\,°\text{F}$,

$$\alpha(-14) = 6.00 \times 10^{-6} + 0.007 \times 10^{-6}(-14)$$
$$= 5.902 \times 10^{-6} \text{ in/in/°F}$$

Example 2

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D\alpha\Delta T$$

where

$D =$ original diameter (in.)

$\alpha =$ coefficient of thermal expansion at average temperature (in/in/°F)

The trunnion is cooled from $80°\text{F}$ to $-108°\text{F}$, giving the average temperature as $-14°\text{F}$. The table of the coefficient of thermal expansion vs. temperature data is given in Table 2.

Table 2 Thermal expansion coefficient as a function of temperature.

Temperature, T $(°\text{F})$	Thermal Expansion Coefficient, α (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
–60	5.58×10^{-6}
–160	4.72×10^{-6}
–260	3.58×10^{-6}
–340	2.45×10^{-6}

If the coefficient of thermal expansion needs to be calculated at the average temperature of $-14°\text{F}$, determine the value of the coefficient of thermal expansion at $T = -14°\text{F}$ using the direct method of interpolation and a first order polynomial.

Solution

For second order polynomial interpolation (also called quadratic interpolation), we choose the coefficient of thermal expansion given by

$$\alpha(T) = a_0 + a_1T + a_2T^2$$

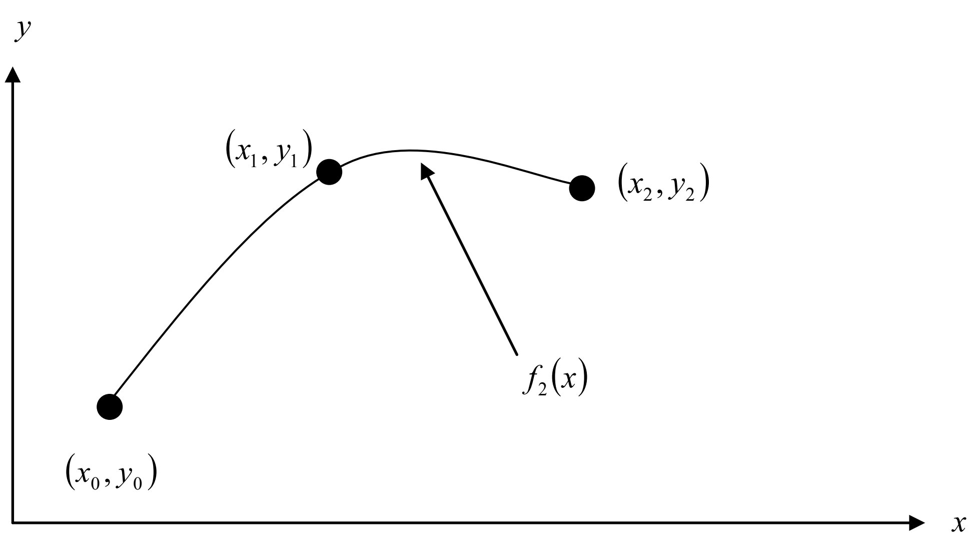

Figure 4 Quadratic interpolation.

Since we want to find the coefficient of thermal expansion at $T = -14\,°\text{F}$, and we are using a second order polynomial, we need to choose the three data points that are closest to $T = -14\,°\text{F}$ that also bracket $T = -14\,°\text{F}$ to evaluate it. These three points are $T_0 = 80\ °\text{F}$, $T_1 = 0\,°\text{F}$ and $T_2 = -60\,°\text{F}$.

Then

$$T_0 = 80, \quad \alpha(T_0) = 6.47 \times 10^{-6}$$
$$T_1 = 0, \quad \alpha(T_1) = 6.00 \times 10^{-6}$$
$$T_2 = -60, \quad \alpha(T_2) = 5.58 \times 10^{-6}$$

gives

$$\alpha(80) = a_0 + a_1(80) + a_2(80)^2 = 6.47 \times 10^{-6}$$
$$\alpha(0) = a_0 + a_1(0) + a_2(0)^2 = 6.00 \times 10^{-6}$$
$$\alpha(-60) = a_0 + a_1(-60) + a_2(-60)^2 = 5.58 \times 10^{-6}$$

Writing the three equations in matrix form, we have

$$\begin{bmatrix} 1 & 80 & 6400 \\ 1 & 0 & 0 \\ 1 & -60 & 3600 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6.47 \times 10^{-6} \\ 6.00 \times 10^{-6} \\ 5.58 \times 10^{-6} \end{bmatrix}$$

Solving the above three equations gives

$$a_0 = 6.00 \times 10^{-6}$$
$$a_1 = 6.5179 \times 10^{-9}$$
$$a_2 = -8.0357 \times 10^{-12}$$

Hence

$$\alpha(T) = 6.00 \times 10^{-6} + 6.5179 \times 10^{-9} T - 8.0357 \times 10^{-12} T^2, \ -60 \le T \le 80$$

At $T = -14\,°\text{F}$,

$$\alpha(-14) = 6.00 \times 10^{-6} + 6.5179 \times 10^{-9}(-14) - 8.0357 \times 10^{-12}(-14)^2$$
$$= 5.9072 \times 10^{-6} \text{ in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left| \frac{5.9072 \times 10^{-6} - 5.902 \times 10^{-6}}{5.9072 \times 10^{-6}} \right| \times 100$$
$$= 0.087605\%$$

Example 3

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D \alpha \Delta T$$

where

$D =$ original diameter (in.)

$\alpha =$ coefficient of thermal expansion at average temperature (in/in/°F)

The trunnion is cooled from $80°\text{F}$ to $-108°\text{F}$, giving the average temperature as $-14°\text{F}$. The table of the coefficient of thermal expansion vs. temperature data is given in Table 3.

Table 3 Thermal expansion coefficient as a function of temperature.

Temperature, T $(°\text{F})$	Thermal Expansion Coefficient, α (in/in/°F)
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–160	4.72×10^{-6}
–260	3.58×10^{-6}
–340	2.45×10^{-6}

a) If the coefficient of thermal expansion needs to be calculated at the average temperature of $-14°\text{F}$, determine the value of the coefficient of thermal expansion at $T=-14°\text{F}$ using the direct method of interpolation and a first order polynomial. Find the absolute relative approximate error for the third order polynomial approximation.

b) The actual reduction in diameter is given by

$$\Delta D = D\int_{T_r}^{T_f} \alpha dT$$

where $T_r =$ room temperature $(°\text{F})$

$T_f =$ temperature of cooling medium $(°\text{F})$

Since

$$T_r = 80°\text{F}$$

$$T_f = -108°\text{F}$$

$$\Delta D = D\int_{80}^{-108} \alpha dT$$

Find out the percentage difference in the reduction in the diameter by the above integral formula and the result using the thermal expansion coefficient from part (a).

Solution

a) For third order polynomial interpolation (also called cubic interpolation), we choose the coefficient of thermal expansion given by

$$\alpha(T) = a_0 + a_1T + a_2T^2 + a_3T^3$$

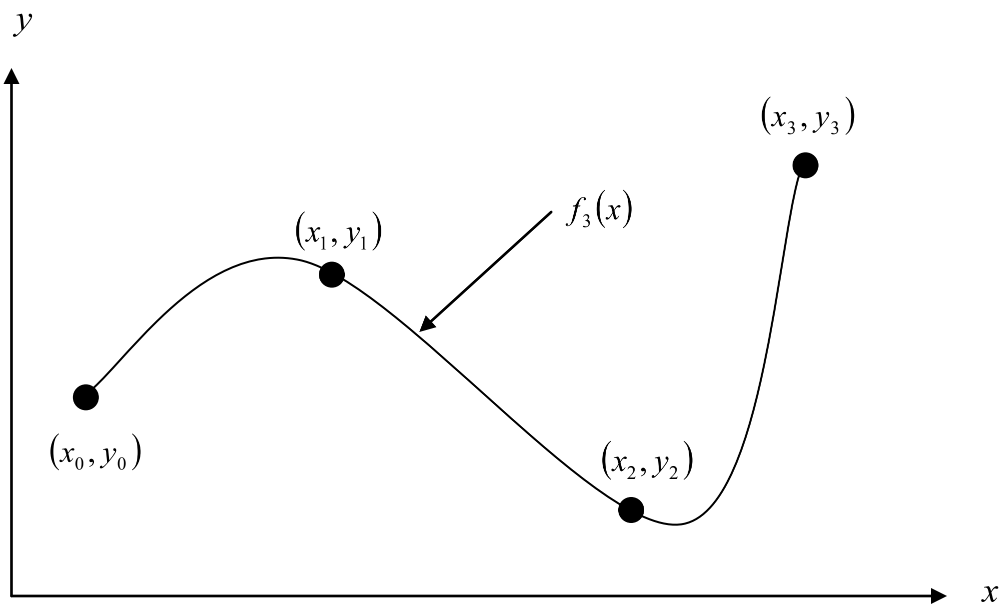

Figure 5 Cubic interpolation.

Since we want to find the coefficient of thermal expansion at $T = -14\,°\text{F}$, and we are using a third order polynomial, we need to choose the four data points closest to $T = -14\,°\text{F}$ that also bracket $T = -14\,°\text{F}$ to evaluate it. Then the four points are $\text{T}_0 = 80\,°\text{F}$, $\text{T}_1 = 0\,°\text{F}$, $T_2 = -60\,°\text{F}$ and $T_3 = -160\,°\text{F}$.

$$T_0 = 80, \quad \alpha(T_0) = 6.47 \times 10^{-6}$$
$$T_1 = 0, \quad \alpha(T_1) = 6.00 \times 10^{-6}$$
$$T_2 = -60, \quad \alpha(T_2) = 5.58 \times 10^{-6}$$
$$T_3 = -160, \quad \alpha(T_3) = 4.72 \times 10^{-6}$$

gives

$$\alpha(80) = a_0 + a_1(80) + a_2(80)^2 + a_3(80)^3 = 6.47 \times 10^{-6}$$
$$\alpha(0) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 = 6.00 \times 10^{-6}$$
$$\alpha(-60) = a_0 + a_1(-60) + a_2(-60)^2 + a_3(-60)^3 = 5.58 \times 10^{-6}$$
$$\alpha(-160) = a_0 + a_1(-160) + a_2(-160)^2 + a_3(-160)^3 = 4.72 \times 10^{-6}$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 80 & 6400 & 5.12 \times 10^5 \\ 1 & 0 & 0 & 0 \\ 1 & -60 & 3600 & -2.16 \times 10^5 \\ 1 & -160 & 25600 & -4.096 \times 10^6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 6.47 \times 10^{-6} \\ 6.00 \times 10^{-6} \\ 5.58 \times 10^{-6} \\ 4.72 \times 10^{-6} \end{bmatrix}$$

Solving the above four equations gives

$$a_0 = 6.00 \times 10^{-6}$$

$$a_1 = 6.4786\times 10^{-9}$$
$$a_2 = -8.1994\times 10^{-12}$$
$$a_3 = 8.1845\times 10^{-15}$$

Hence

$$\alpha(T) = a_0 + a_1 T + a_2 T^2 + a_3 T^3$$
$$= 6.00\times 10^{-6} + 6.4786\times 10^{-9} T - 8.1994\times 10^{-12} T^2 + 8.1845\times 10^{-15} T^3, \quad -160 \le T \le 80$$
$$\alpha(-14) = 6.00\times 10^{-6} + 6.4786\times 10^{-9}(-14) - 8.1994\times 10^{-12}(-14)^2 + 8.1845\times 10^{-15}(-14)^3$$
$$= 5.9077\times 10^{-6} \text{ in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a| = \left|\frac{5.9077\times 10^{-6} - 5.9072\times 10^{-6}}{5.9077\times 10^{-6}}\right|\times 100$$
$$= 0.0083867\%$$

b) In finding the percentage difference in the reduction in diameter, we can rearrange the integral formula to

$$\frac{\Delta D}{D} = \int_{T_r}^{T_f} \alpha dT$$

and since we know from part (a) that

$$\alpha(T) = 6.00\times 10^{-6} + 6.4786\times 10^{-9} T - 8.1994\times 10^{-12} T^2 + 8.1845\times 10^{-15} T^3, \quad -160 \le T \le 80$$

we see that we can use the integral formula in the range from $T_f = -108$°F to $T_r = 80$°F

Therefore,

$$\frac{\Delta D}{D} = \int_{T_r}^{T_f} \alpha dT$$
$$= \int_{80}^{-108} \left(6.00\times 10^{-6} + 6.4786\times 10^{-9} T - 8.1994\times 10^{-12} T^2 + 8.1845\times 10^{-15} T^3\right) dT$$
$$= \left[6.00\times 10^{-6} T + 6.4786\times 10^{-9}\frac{T^2}{2} - 8.1994\times 10^{-12}\frac{T^3}{3} + 8.1845\times 10^{-15}\frac{T^4}{4}\right]_{80}^{-108}$$
$$= -1105.9\times 10^{-6}$$

So $\frac{\Delta D}{D} = -1105.9\times 10^{-6}$ in/in using the actual reduction in diameter integral formula. If we use the average value for the coefficient of thermal expansion from part (a), we get

$$\frac{\Delta D}{D} = \alpha \Delta T$$
$$= \alpha(T_f - T_r)$$
$$= 5.9077\times 10^{-6}(-108 - 80)$$
$$= -1110.6\times 10^{-6}$$

and $\frac{\Delta D}{D} = -1110.6 \times 10^{-6}$ in/in using the average value of the coefficient of thermal expansion using a third order polynomial. Considering the integral to be the more accurate calculation, the percentage difference would be

$$\left|\in_a\right| = \left|\frac{-1105.9\times10^{-6} - \left(-1110.6\times10^{-6}\right)}{-1105.9\times10^{-6}}\right|\times100$$

$$= 0.42775\%$$

INTERPOLATION	
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