Chapter 05.04
Lagrangian Interpolation

After reading this chapter, you should be able to:

1. *derive Lagrangian method of interpolation,*
2. *solve problems using Lagrangian method of interpolation, and*
3. *use Lagrangian interpolants to find derivatives and integrals of discrete functions.*

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

(A) evaluate,
(B) differentiate, and
(C) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ data points. One of the methods used to find this polynomial is called the Lagrangian method of interpolation. Other methods include Newton's divided difference polynomial method and the direct method. We discuss the Lagrangian method in this chapter.

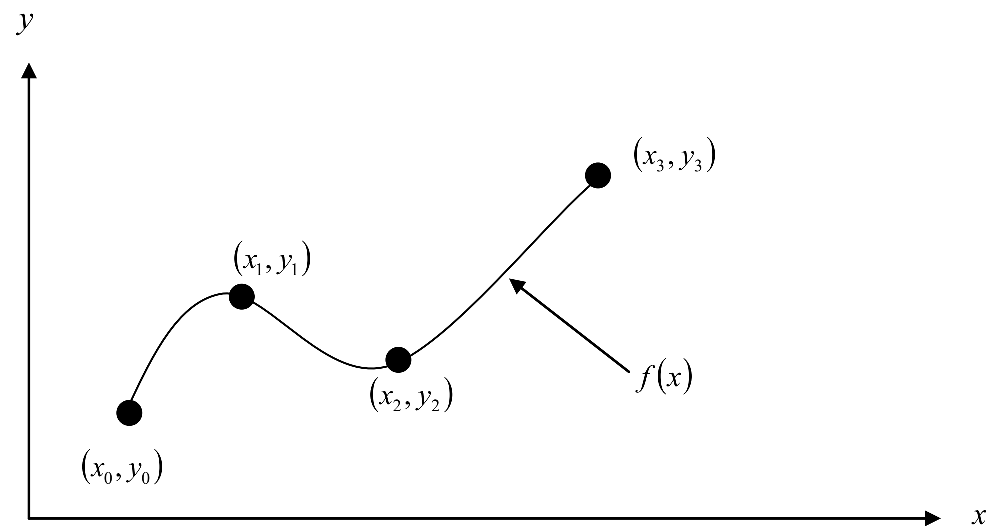

Figure 1 Interpolation of discrete data.

The Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $n+1$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j = i$ omitted. The application of Lagrangian interpolation will be clarified using an example.

Example 1

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D\alpha\Delta T$$

where

$D =$ original diameter (in.)

$\alpha =$ coefficient of thermal expansion at average temperature (in/in/°F)

The trunnion is cooled from $80°\text{F}$ to $-108°\text{F}$, giving the average temperature as $-14°\text{F}$. The table of the coefficient of thermal expansion vs. temperature data is given in Table 1.

Table 1 Thermal expansion coefficient as a function of temperature.

Temperature, T $(°\text{F})$	Thermal Expansion Coefficient, α (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
–60	5.58×10^{-6}
–160	4.72×10^{-6}
–260	3.58×10^{-6}
–340	2.45×10^{-6}

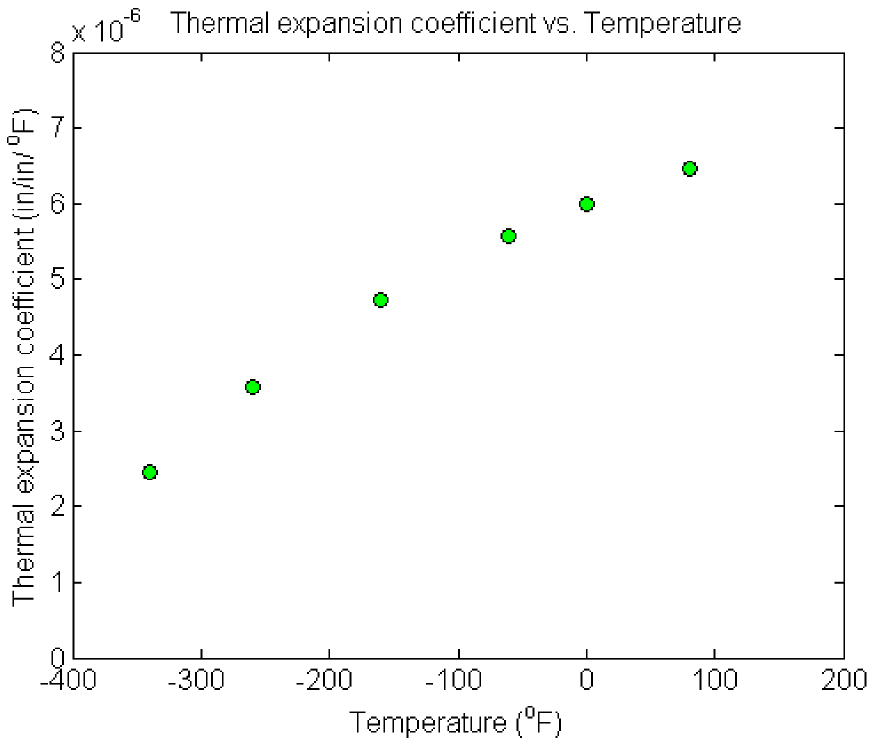

Figure 2 Thermal expansion coefficient vs. temperature.

If the coefficient of thermal expansion needs to be calculated at the average temperature of $-14°\text{F}$, determine the value of the coefficient of thermal expansion at $T=-14°\text{F}$ using a first order Lagrange polynomial.

Solution

For first order Lagrange polynomial interpolation (also called linear interpolation), the coefficient of thermal expansion is given by

$$\begin{aligned}\alpha(T) &= \sum_{i=0}^{1} L_i(T)\alpha(T_i)\\ &= L_0(T)\alpha(T_0)+L_1(T)\alpha(T_1)\end{aligned}$$

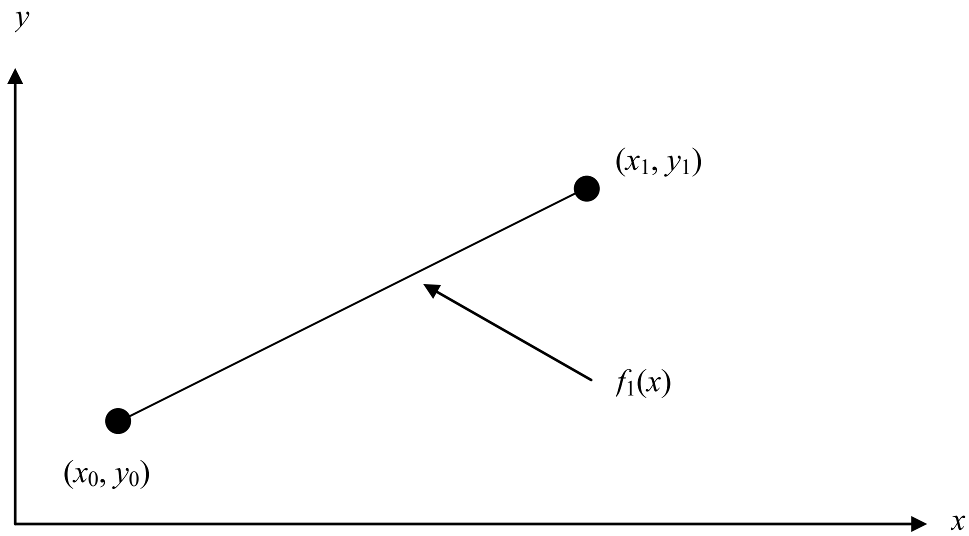

Figure 3 Linear interpolation.

Since we want to find the coefficient of thermal expansion at $T = -14\,°\text{F}$, we choose two data points that are closest to $T = -14\,°\text{F}$ and that also bracket $T = -14\,°\text{F}$. The two points are $T_0 = 0$ and $T_1 = -60\,°\text{F}$.

Then

$$T_0 = 0,\ \alpha(T_0) = 6.00\times 10^{-6}$$
$$T_1 = -60,\ \alpha(T_1) = 5.58\times 10^{-6}$$

gives

$$L_0(T) = \prod_{\substack{j=0\\ j\neq 0}}^{1} \frac{T - T_j}{T_0 - T_j}$$
$$= \frac{T - T_1}{T_0 - T_1}$$
$$L_1(T) = \prod_{\substack{j=0\\ j\neq 1}}^{1} \frac{T - T_j}{T_1 - T_j}$$
$$= \frac{T - T_0}{T_1 - T_0}$$

Hence

$$\alpha(T) = \frac{T - T_1}{T_0 - T_1}\alpha(T_0) + \frac{T - T_0}{T_1 - T_0}\alpha(T_1)$$
$$= \frac{T + 60}{0 + 60}(6.00\times 10^{-6}) + \frac{T - 0}{-60 - 0}(5.58\times 10^{-6}),\ -60 \le T \le 0$$
$$\alpha(-14) = \frac{-14 + 60}{0 + 60}(6.00\times 10^{-6}) + \frac{-14 - 0}{-60 - 0}(5.58\times 10^{-6})$$

$$= 0.76667(6.00\times10^{-6}) + 0.23333(5.58\times10^{-6})$$
$$= 5.902\times10^{-6} \text{ in/in/°F}$$

You can see that $L_0(T) = 0.76667$ and $L_1(T) = 0.23333$ are like weightages given to the coefficients of thermal expansion at $T = 0$ and $T = -60\,°\text{F}$ to calculate the coefficient of thermal expansion at $T = -14\,°\text{F}$.

Example 2

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D\alpha\Delta T$$

where

$D =$ original diameter (in.)

$\alpha =$ coefficient of thermal expansion at average temperature (in/in/°F)

The trunnion is cooled from $80\,°\text{F}$ to $-108\,°\text{F}$, giving the average temperature as $-14\,°\text{F}$. The table of the coefficient of thermal expansion vs. temperature data is given in Table 2.

Table 2 Thermal expansion coefficient as a function of temperature.

Temperature, T(°F)	Thermal Expansion Coefficient, α(in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
–60	5.58×10^{-6}
–160	4.72×10^{-6}
–260	3.58×10^{-6}
–340	2.45×10^{-6}

If the coefficient of thermal expansion needs to be calculated at the average temperature of $-14\,°\text{F}$, determine the value of the coefficient of thermal expansion at $T = -14\,°\text{F}$ using a second order Lagrangian polynomial. Find the absolute relative approximate error for the second order polynomial interpolation.

Solution

For second order Lagrange polynomial interpolation (also called quadratic interpolation), the coefficient of thermal expansion given by

$$\alpha(T) = \sum_{i=0}^{2} L_i(T)\alpha(T_i)$$
$$= L_0(T)\alpha(T_0) + L_1(T)\alpha(T_1) + L_2(T)\alpha(T_2)$$

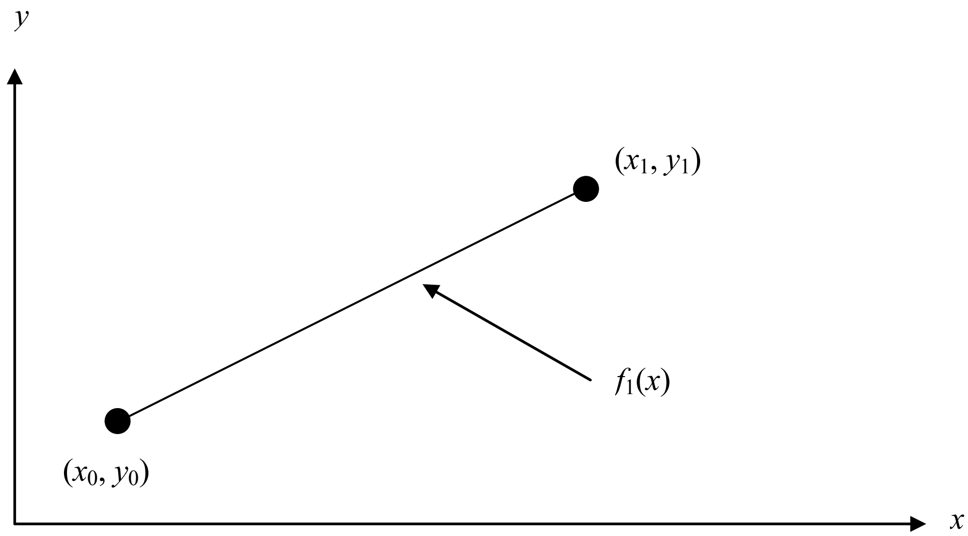

Figure 4 Quadratic interpolation.

Since we want to find the coefficient of thermal expansion at $T = -14\,°\text{F}$, we need to choose data points that are closest to $T = -14\,°\text{F}$ that also bracket $T = -14\,°\text{F}$ to evaluate it. The three points are $T_0 = 80\,°\text{F}$, $T_1 = 0$ and $T_2 = -60\,°\text{F}$.

$$T_0 = 80,\ \ \alpha(T_0) = 6.47\times10^{-6}$$
$$T_1 = 0,\ \ \alpha(T_1) = 6.00\times10^{-6}$$
$$T_2 = -60,\ \ \alpha(T_2) = 5.58\times10^{-6}$$

gives

$$L_0(T) = \prod_{\substack{j=0\\j\neq 0}}^{2} \frac{T - T_j}{T_0 - T_j}$$
$$= \left(\frac{T - T_1}{T_0 - T_1}\right)\left(\frac{T - T_2}{T_0 - T_2}\right)$$
$$L_1(T) = \prod_{\substack{j=0\\j\neq 1}}^{2} \frac{T - T_j}{T_1 - T_j}$$
$$= \left(\frac{T - T_0}{T_1 - T_0}\right)\left(\frac{T - T_2}{T_1 - T_2}\right)$$
$$L_2(T) = \prod_{\substack{j=0\\j\neq 2}}^{2} \frac{T - T_j}{T_2 - T_j}$$
$$= \left(\frac{T - T_0}{T_2 - T_0}\right)\left(\frac{T - T_1}{T_2 - T_1}\right)$$

Hence

$$\alpha(T) = \left(\frac{T-T_1}{T_0-T_1}\right)\left(\frac{T-T_2}{T_0-T_2}\right)\alpha(T_0) + \left(\frac{T-T_0}{T_1-T_0}\right)\left(\frac{T-T_2}{T_1-T_2}\right)\alpha(T_1) + \left(\frac{T-T_0}{T_2-T_0}\right)\left(\frac{T-T_1}{T_2-T_1}\right)\alpha(T_2),$$

$$T_0 \le T \le T_2$$

$$\alpha(-14) = \frac{(-14-0)(-14+60)}{(80-0)(80+60)}(6.47\times10^{-6}) + \frac{(-14-80)(-14+60)}{(0-80)(0+60)}(6.00\times10^{-6})$$

$$+\frac{(-14-80)(-14-0)}{(-60-80)(-60-0)}(5.58\times10^{-6})$$

$$= (-0.0575)(6.47\times10^{-6}) + (0.90083)(6.00\times10^{-6}) + (0.15667)(5.58\times10^{-6})$$

$$= 5.9072\times10^{-6} \text{ in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{5.9072\times10^{-6} - 5.902\times10^{-6}}{5.9072\times10^{-6}}\right| \times 100$$

$$= 0.087605\%$$

Example 3

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D\alpha\Delta T$$

where

$D =$ original diameter (in.)

$\alpha =$ coefficient of thermal expansion at average temperature (in/in/°F)

The trunnion is cooled from $80°\text{F}$ to $-108°\text{F}$, giving the average temperature as $-14°\text{F}$. The table of the coefficient of thermal expansion vs. temperature data is given in Table 3.

Table 3 Thermal expansion coefficient as a function of temperature.

Temperature, T(°F)	Thermal Expansion Coefficient, α(in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
–60	5.58×10^{-6}
–160	4.72×10^{-6}
–260	3.58×10^{-6}
–340	2.45×10^{-6}

a) If the coefficient of thermal expansion needs to be calculated at the average temperature of $-14°\text{F}$, determine the value of the coefficient of thermal expansion at $T = -14°\text{F}$ a third order Lagrange polynomial. Find the absolute relative approximate error for the third order polynomial approximation.
b) The actual reduction in diameter is given by

$$\Delta D = D\int_{T_r}^{T_f} \alpha dT$$

where T_r = room temperature $(°\mathrm{F})$

T_f = temperature of cooling medium $(°\mathrm{F})$

Since

$$T_r = 80\,°\mathrm{F}$$
$$T_f = -108\,°\mathrm{F}$$
$$\Delta D = D\int_{80}^{-108} \alpha dT$$

Find out the percentage difference in the reduction in the diameter by the above integral formula and the result using the thermal expansion coefficient from part (a).

Solution

a) For third order Lagrange polynomial interpolation (also called cubic interpolation), the coefficient of thermal expansion is given by

$$\begin{aligned}\alpha(T) &= \sum_{i=0}^{3} L_i(T)\alpha(T_i)\\ &= L_0(T)\alpha(T_0) + L_1(T)\alpha(T_1) + L_2(T)\alpha(T_2) + L_3(T)\alpha(T_3)\end{aligned}$$

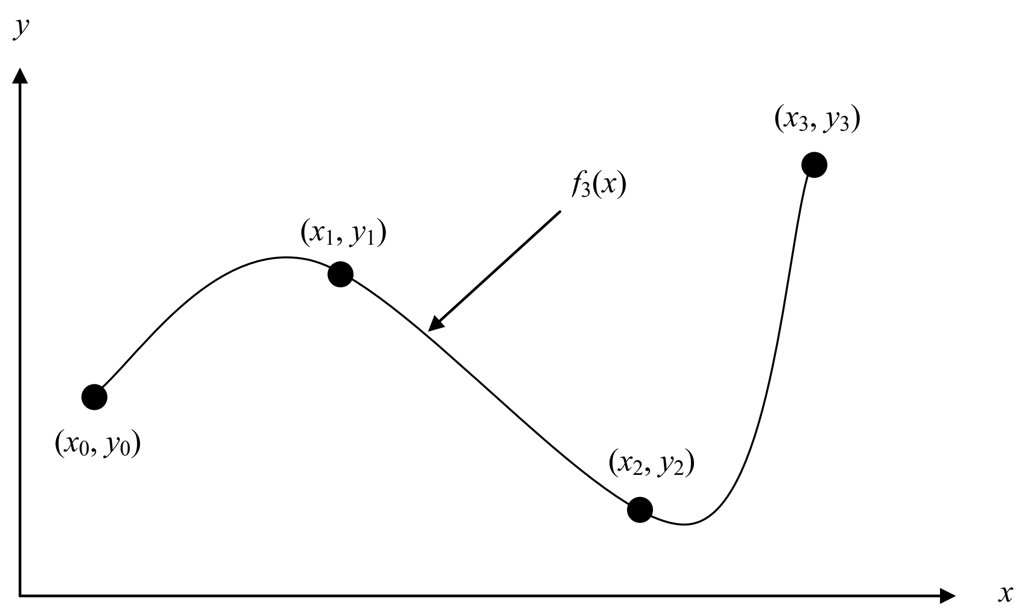

Figure 5 Cubic interpolation.

Since we want to find the coefficient of thermal expansion at $T = -14\,°\mathrm{F}$, and we are using a third order polynomial, we need to choose the four points closest to $T = -14\,°\mathrm{F}$ that also bracket $T = -14\,°\text{F}$ to evaluate it. The four points are $T_0 = 80\,°\text{F}$, $T_1 = 0$, $T_2 = -60\,°\text{F}$ and $T_3 = -160\,°\text{F}$.

Then

$$T_0 = 80,\ \ \alpha(T_0) = 6.47 \times 10^{-6}$$
$$T_1 = 0,\ \ \alpha(T_1) = 6.00 \times 10^{-6}$$
$$T_2 = -60,\ \ \alpha(T_2) = 5.58 \times 10^{-6}$$
$$T_3 = -160,\ \ \alpha(T_3) = 4.72 \times 10^{-6}$$

gives

$$L_0(T) = \prod_{\substack{j=0 \\ j \neq 0}}^{3} \frac{T - T_j}{T_0 - T_j}$$
$$= \left(\frac{T - T_1}{T_0 - T_1}\right)\left(\frac{T - T_2}{T_0 - T_2}\right)\left(\frac{T - T_3}{T_0 - T_3}\right)$$

$$L_1(T) = \prod_{\substack{j=0 \\ j \neq 1}}^{3} \frac{T - T_j}{T_1 - T_j}$$
$$= \left(\frac{T - T_0}{T_1 - T_0}\right)\left(\frac{T - T_2}{T_1 - T_2}\right)\left(\frac{T - T_3}{T_1 - T_3}\right)$$

$$L_2(T) = \prod_{\substack{j=0 \\ j \neq 2}}^{3} \frac{T - T_j}{T_2 - T_j}$$
$$= \left(\frac{T - T_0}{T_2 - T_0}\right)\left(\frac{T - T_1}{T_2 - T_1}\right)\left(\frac{T - T_3}{T_2 - T_3}\right)$$

$$L_3(T) = \prod_{\substack{j=0 \\ j \neq 3}}^{3} \frac{T - T_j}{T_3 - T_j}$$
$$= \left(\frac{T - T_0}{T_3 - T_0}\right)\left(\frac{T - T_1}{T_3 - T_1}\right)\left(\frac{T - T_2}{T_3 - T_2}\right)$$

Hence

$$\alpha(T) = \left(\frac{T - T_1}{T_0 - T_1}\right)\left(\frac{T - T_2}{T_0 - T_2}\right)\left(\frac{T - T_3}{T_0 - T_3}\right)\alpha(T_0) + \left(\frac{T - T_0}{T_1 - T_0}\right)\left(\frac{T - T_2}{T_1 - T_2}\right)\left(\frac{T - T_3}{T_1 - T_3}\right)\alpha(T_1)$$
$$+ \left(\frac{T - T_0}{T_2 - T_0}\right)\left(\frac{T - T_1}{T_2 - T_1}\right)\left(\frac{T - T_3}{T_2 - T_3}\right)\alpha(T_2) + \left(\frac{T - T_0}{T_3 - T_0}\right)\left(\frac{T - T_1}{T_3 - T_1}\right)\left(\frac{T - T_2}{T_3 - T_2}\right)\alpha(T_3)$$
$$T_0 \le T \le T_3$$

$$\alpha(-14) = \frac{(-14-0)(-14+60)(-14+160)}{(80-0)(80+60)(80+160)}(6.47\times10^{-6})$$
$$+\frac{(-14-80)(-14+60)(-14+160)}{(0-80)(0+60)(0+160)}(6.00\times10^{-6})$$
$$+\frac{(-14-80)(-14-0)(-14+160)}{(-60-80)(-60-0)(-60+160)}(5.58\times10^{-6})$$
$$+\frac{(-14-80)(-14-0)(-14+60)}{(-160-80)(-160-0)(-160+60)}(4.72\times10^{-6})$$
$$= (-0.034979)(6.47\times10^{-6}) + (0.82201)(6.00\times10^{-6}) + (0.22873)(5.58\times10^{-6})$$
$$+ (-0.015765)(4.72\times10^{-6})$$
$$= 5.9077\times10^{-6}\ \text{in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a| = \left|\frac{5.9077\times10^{-6} - 5.9072\times10^{-6}}{5.9077\times10^{-6}}\right|\times100$$
$$= 0.0083867\%$$

b) In finding the percentage difference in the reduction in diameter, we can rearrange the integral formula to

$$\frac{\Delta D}{D} = \int_{T_r}^{T_f} \alpha dT$$

and since we know from part (a) that

$$\alpha(T) = \frac{(T-0)(T+60)(T+160)}{(80-0)(80+60)(80+160)}(6.47\times10^{-6})$$
$$+\frac{(T-80)(T+60)(T+160)}{(0-80)(0+60)(0+160)}(6.00\times10^{-6})$$
$$+\frac{(T-80)(T-0)(T+160)}{(-60-80)(-60-0)(-60+160)}(5.58\times10^{-6})$$
$$+\frac{(T-80)(T-0)(T+60)}{(-160-80)(-160-0)(-160+60)}(4.72\times10^{-6}), \qquad -160 \le T \le 80$$

Combining like terms, we get

$$\alpha(T) = 6.00\times10^{-6} + 6.4786\times10^{-9}T - 8.1994\times10^{-12}T^2 + 8.1845\times10^{-15}T^3,$$
$$-160 \le T \le 80$$

We see that we can use the integral formula in the range from $T_f = -108$°F to $T_r = 80$°F

Therefore,

$$\frac{\Delta D}{D} = \int_{T_r}^{T_f} \alpha dT$$

$$= \int_{80}^{-108} (6.00\times10^{-6} + 6.4786\times10^{-9}T - 8.1994\times10^{-12}T^2 + 8.1845\times10^{-15}T^3)dT$$

$$= \left[6.00\times10^{-6}T + 6.4786\times10^{-9}\frac{T^2}{2} - 8.1994\times10^{-12}\frac{T^3}{3} + 8.1845\times10^{-15}\frac{T^4}{4}\right]_{80}^{-108}$$

$$= -1105.9\times10^{-6}$$

So $\frac{\Delta D}{D} = -1105.9\times10^{-6}$ in/in using the actual reduction in diameter integral formula. If we use the average value for the coefficient of thermal expansion from part (a), we get

$$\frac{\Delta D}{D} = \alpha\Delta T$$

$$= \alpha(T_f - T_r)$$

$$= 5.9077\times10^{-6}(-108-80)$$

$$= -1110.6\times10^{-6}$$

and $\frac{\Delta D}{D} = -1110.6\times10^{-6}$ in/in using the average value of the coefficient of thermal expansion using a third order polynomial. Considering the integral to be the more accurate calculation, the percentage difference would be

$$|\in_a| = \left|\frac{-1105.9\times10^{-6} + 1110.6\times10^{-6}}{-1105.9\times10^{-6}}\right|\times100$$

$$= 0.42775\%$$

INTERPOLATION	
Topic	Lagrange Interpolation
Summary	Textbook notes on the Lagrangian method of interpolation
Major	Mechanical Engineering
Authors	Autar Kaw, Michael Keteltas
Last Revised	November 17, 2012
Web Site	http://numericalmethods.eng.usf.edu