Chapter 05.03
Newton's Divided Difference Interpolation

After reading this chapter, you should be able to:

1. *derive Newton's divided difference method of interpolation,*
2. *apply Newton's divided difference method of interpolation, and*
3. *apply Newton's divided difference method interpolants to find derivatives and integrals.*

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

(A) evaluate,
(B) differentiate, and
(C) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ points. One of the methods of interpolation is called Newton's divided difference polynomial method. Other methods include the direct method and the Lagrangian interpolation method. We will discuss Newton's divided difference polynomial method in this chapter.

Newton's Divided Difference Polynomial Method

To illustrate this method, linear and quadratic interpolation is presented first. Then, the general form of Newton's divided difference polynomial method is presented. To illustrate the general form, cubic interpolation is shown in Figure 1.

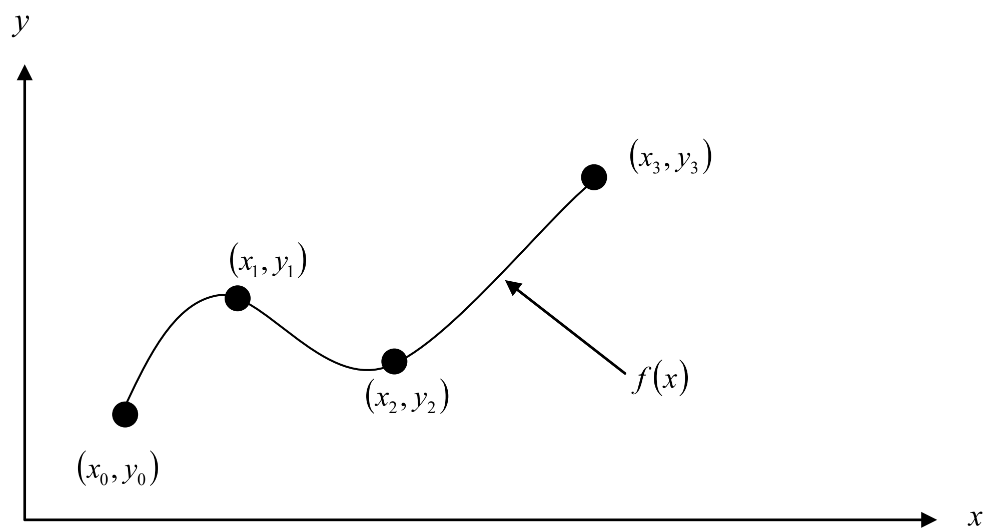

Figure 1 Interpolation of discrete data.

Linear Interpolation

Given (x_0, y_0) and (x_1, y_1), fit a linear interpolant through the data. Noting $y = f(x)$ and $y_1 = f(x_1)$, assume the linear interpolant $f_1(x)$ is given by (Figure 2)

$$f_1(x) = b_0 + b_1(x - x_0)$$

Since at $x = x_0$,

$$f_1(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) = b_0$$

and at $x = x_1$,

$$f_1(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0)$$
$$= f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So

$$b_0 = f(x_0)$$
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

giving the linear interpolant as

$$f_1(x) = b_0 + b_1(x - x_0)$$
$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

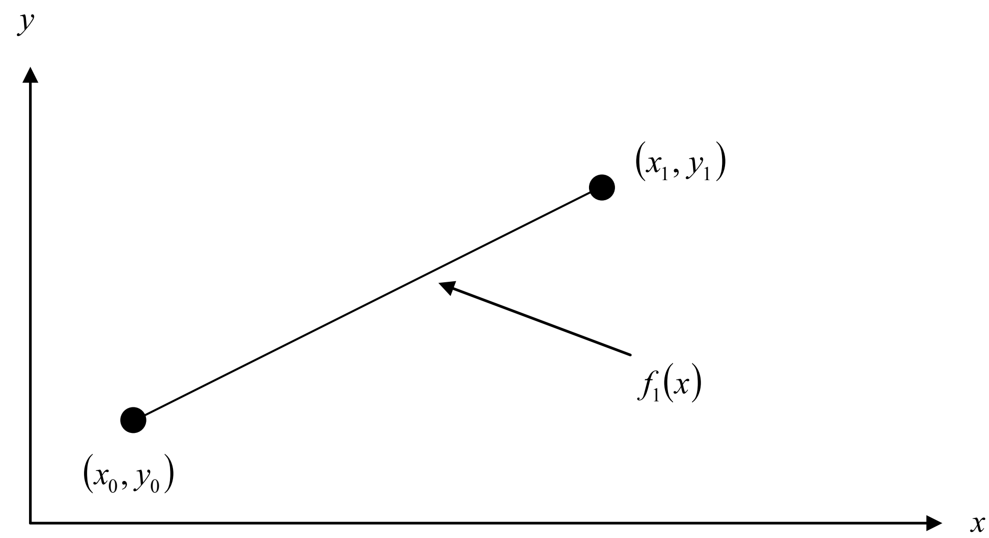

Figure 2 Linear interpolation.

Example 1

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D\alpha\Delta T$$

where

$D =$ original diameter (in.)

$\alpha =$ coefficient of thermal expansion at average temperature (in/in/°F)

The trunnion is cooled from $80°\text{F}$ to $-108°\text{F}$, giving the average temperature as $-14°\text{F}$. The table of the coefficient of thermal expansion vs. temperature data is given in Table 1.

Table 1 Thermal expansion coefficient as a function of temperature.

Temperature, T (°F)	Thermal Expansion Coefficient, α (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
–60	5.58×10^{-6}
–160	4.72×10^{-6}
–260	3.58×10^{-6}
–340	2.45×10^{-6}

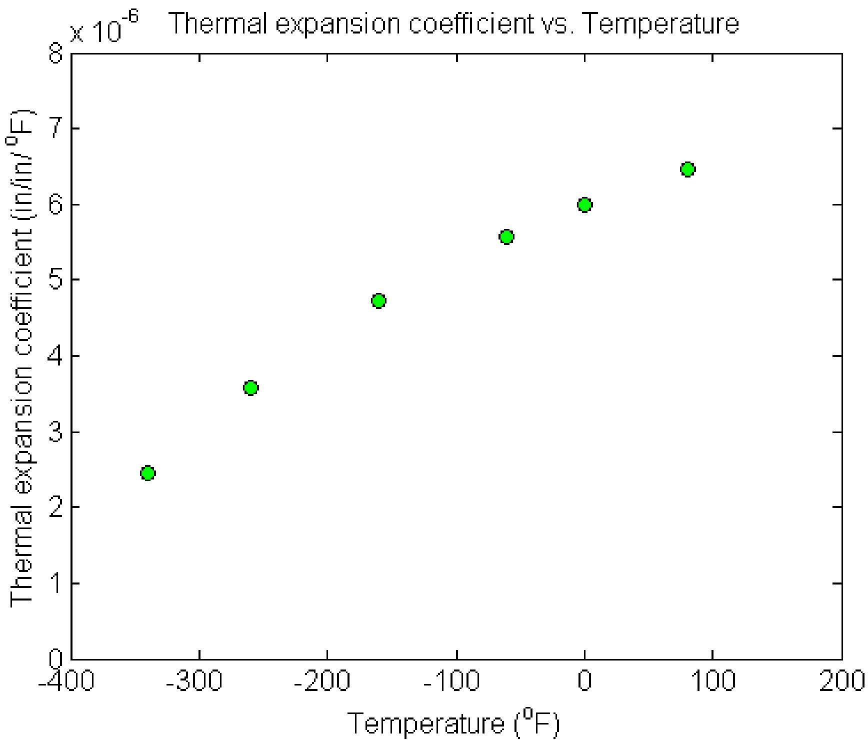

Figure 3 Thermal expansion coefficient vs. temperature.

Determine the value of the coefficient of thermal expansion at $T = -14\,°\mathrm{F}$ using Newton's divided difference method of interpolation and a first order polynomial.

Solution

For linear interpolation, the coefficient of thermal expansion is given by

$$\alpha(T) = b_0 + b_1(T - T_0)$$

Since we want to find the coefficient of thermal expansion at $T = -14$ and we are using a first order polynomial, we need to choose the two data points that are closest to $T = -14$ that also bracket $T = -14$ to evaluate it. The two points are $T_0 = 0$ and $T_1 = -60$.

Then

$$T_0 = 0,\ \alpha(T_0) = 6.00 \times 10^{-6}$$

$$T_1 = -60,\ \alpha(T_1) = 5.58 \times 10^{-6}$$

gives

$$b_0 = \alpha(T_0)$$

$$= 6.00 \times 10^{-6}$$

$$b_1 = \frac{\alpha(T_1) - \alpha(T_0)}{T_1 - T_0}$$

$$= \frac{5.58 \times 10^{-6} - 6.00 \times 10^{-6}}{-60 - 0}$$

$$= 0.007 \times 10^{-6}$$

Hence

$$\alpha(T) = b_0 + b_1(T - T_0)$$
$$= 6.00\times 10^{-6} + 0.007\times 10^{-6}(T-0), \qquad -60 \le T \le 0$$

At $T = -14$

$$\alpha(-14) = 6.00\times 10^{-6} + 0.007\times 10^{-6}(-14-0)$$
$$= 5.902\times 10^{-6} \text{ in/in/°F}$$

If we expand

$$\alpha(T) = 6.00\times 10^{-6} + 0.007\times 10^{-6}(T-0), \qquad -60 \le T \le 0$$

we get

$$\alpha(T) = 6.00\times 10^{-6} + 0.007\times 10^{-6}T, \qquad -60 \le T \le 0$$

This is the same expression that was obtained with the direct method.

<u>Quadratic Interpolation</u>

Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data. Noting $y = f(x)$, $y_0 = f(x_0)$, $y_1 = f(x_1)$, and $y_2 = f(x_2)$, assume the quadratic interpolant $f_2(x)$ is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

At $x = x_0$,

$$f_2(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1)$$
$$= b_0$$
$$b_0 = f(x_0)$$

At $x = x_1$

$$f_2(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1)$$
$$f(x_1) = f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

At $x = x_2$

$$f_2(x_2) = f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$
$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

Giving

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Hence the quadratic interpolant is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}(x - x_0)(x - x_1)$$

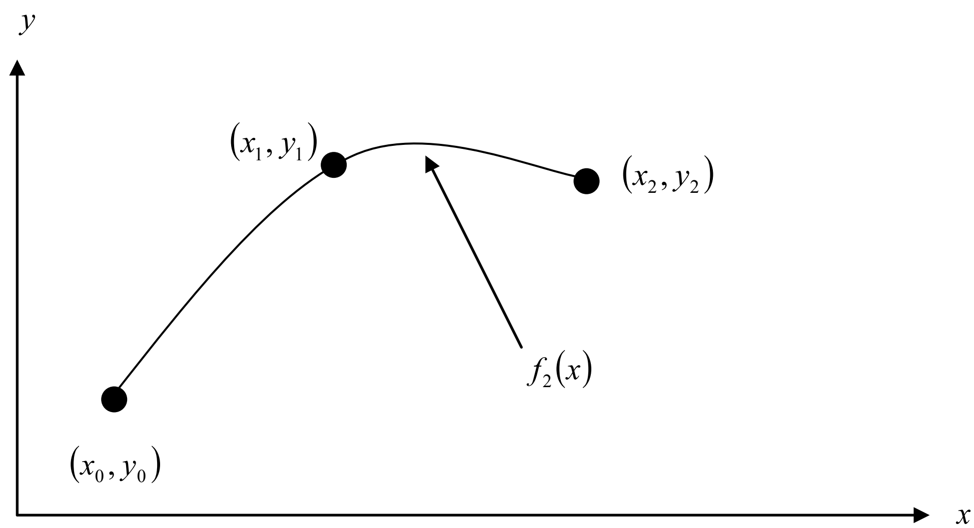

Figure 4 Quadratic interpolation.

Example 2

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D\alpha\Delta T$$

where

$D =$ original diameter (in.)

$\alpha =$ coefficient of thermal expansion at average temperature (in/in/°F)

The trunnion is cooled from $80°\text{F}$ to $-108°\text{F}$, giving the average temperature as $-14°\text{F}$. The table of the coefficient of thermal expansion vs. temperature data is given in Table 2.

Table 2 Thermal expansion coefficient as a function of temperature.

Temperature, T (°F)	Thermal Expansion Coefficient, α (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
–60	5.58×10^{-6}
–160	4.72×10^{-6}
–260	3.58×10^{-6}
–340	2.45×10^{-6}

Determine the value of the coefficient of thermal expansion at $T = -14°\text{F}$ using Newton's divided difference method of interpolation and a second order polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For quadratic interpolation, the coefficient of thermal expansion is given by

$$\alpha(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1)$$

Since we want to find the coefficient of thermal expansion at $T = -14,$ we need to choose the three data points that are closest to $T = -14$ that also bracket $T = -14$ to evaluate it. The three points are $T_0 = 80,$ $T_1 = 0$ and $T_2 = -60$.

Then

$$T_0 = 80, \quad \alpha(T_0) = 6.47 \times 10^{-6}$$

$$T_1 = 0, \quad \alpha(T_1) = 6.00 \times 10^{-6}$$

$$T_2 = -60, \quad \alpha(T_2) = 5.58 \times 10^{-6}$$

gives

$$b_0 = \alpha(T_0)$$

$$= 6.47 \times 10^{-6}$$

$$b_1 = \frac{\alpha(T_1) - \alpha(T_0)}{T_1 - T_0}$$

$$= \frac{6.00 \times 10^{-6} - 6.47 \times 10^{-6}}{0 - 80}$$

$$= 5.875 \times 10^{-9}$$

$$b_2 = \frac{\dfrac{\alpha(T_2) - \alpha(T_1)}{T_2 - T_1} - \dfrac{\alpha(T_1) - \alpha(T_0)}{T_1 - T_0}}{T_2 - T_0}$$

$$= \frac{\dfrac{5.58 \times 10^{-6} - 6.00 \times 10^{-6}}{-60 - 0} - \dfrac{6.00 \times 10^{-6} - 6.47 \times 10^{-6}}{0 - 80}}{-60 - 80}$$

$$= \frac{0.007 \times 10^{-6} - 0.005875 \times 10^{-6}}{-140}$$

$$= -8.0357 \times 10^{-12}$$

Hence

$$\alpha(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1)$$

$$= 6.47 \times 10^{-6} + 5.875 \times 10^{-9}(T - 80) - 8.0357 \times 10^{-12}(T - 80)(T - 0), \ -60 \le T \le 80$$

At $T = -14,$

$$\alpha(-14) = 6.47 \times 10^{-6} + 5.875 \times 10^{-9}(-14 - 80) - 8.0357 \times 10^{-12}(-14 - 80)(-14 - 0)$$

$$= 5.9072 \times 10^{-6} \text{ in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left| \frac{5.9072 \times 10^{-6} - 5.902 \times 10^{-6}}{5.9072 \times 10^{-6}} \right| \times 100$$

$$= 0.087605\%$$

If we expand

$$\alpha(T) = 6.47\times10^{-6} + 5.875\times10^{-9}(T-80) - 8.0357\times10^{-12}(T-80)(T-0),\ -60 \le T \le 80$$

we get

$$\alpha(T) = 6.00\times10^{-6} + 6.5179\times10^{-9}T - 8.0357\times10^{-12}T^2, \quad -60 \le T \le 80$$

This is the same expression that was obtained with the direct method.

General Form of Newton's Divided Difference Polynomial

In the two previous cases, we found linear and quadratic interpolants for Newton's divided difference method. Let us revisit the quadratic polynomial interpolant formula

$$f_2(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1)$$

where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$b_2 = \frac{\dfrac{f(x_2)-f(x_1)}{x_2-x_1} - \dfrac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}$$

Note that b_0, b_1, and b_2 are finite divided differences. b_0, b_1, and b_2 are the first, second, and third finite divided differences, respectively. We denote the first divided difference by

$$f[x_0] = f(x_0)$$

the second divided difference by

$$f[x_1, x_0] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

and the third divided difference by

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2-x_0}$$

$$= \frac{\dfrac{f(x_2)-f(x_1)}{x_2-x_1} - \dfrac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}$$

where $f[x_0]$, $f[x_1, x_0]$, and $f[x_2, x_1, x_0]$ are called bracketed functions of their variables enclosed in square brackets.

Rewriting,

$$f_2(x) = f[x_0] + f[x_1, x_0](x-x_0) + f[x_2, x_1, x_0](x-x_0)(x-x_1)$$

This leads us to writing the general form of the Newton's divided difference polynomial for $n+1$ data points, $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, as

$$f_n(x) = b_0 + b_1(x-x_0) + \ldots. + b_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$
$$\vdots$$
$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots., x_0]$$
$$b_n = f[x_n, x_{n-1}, \ldots., x_0]$$

where the definition of the m^{th} divided difference is

$$b_m = f[x_m, \ldots\ldots.., x_0]$$
$$= \frac{f[x_m, \ldots\ldots.., x_1] - f[x_{m-1}, \ldots\ldots.., x_0]}{x_m - x_0}$$

From the above definition, it can be seen that the divided differences are calculated recursively.

For an example of a third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3),

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$
$$+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

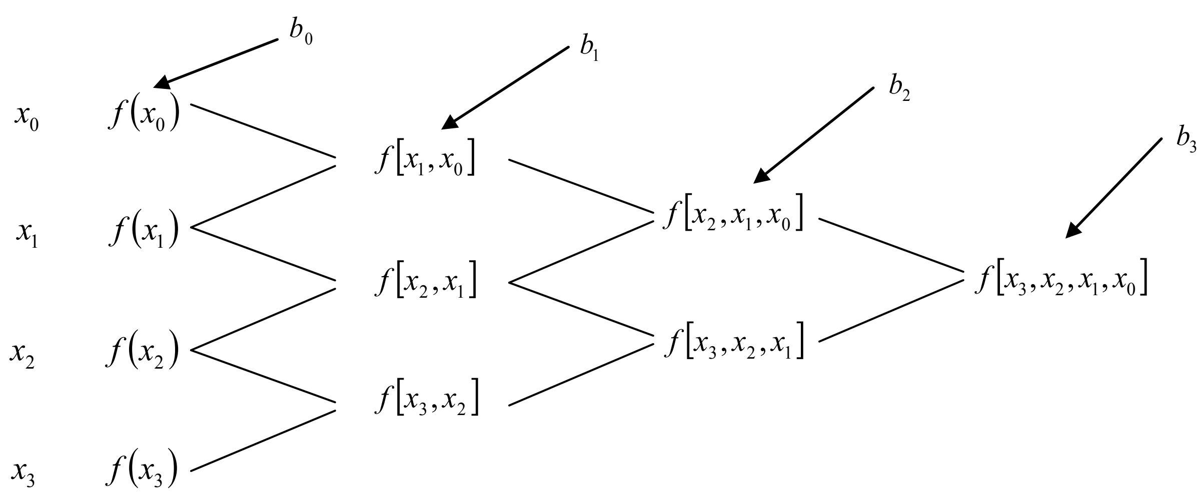

Figure 5 Table of divided differences for a cubic polynomial.

Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Table 3 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t = 16$ seconds with third order polynomial interpolation using Newton's divided difference polynomial method.
b) Using the third order polynomial interpolant for velocity, find the distance covered by the rocket from $t = 11\text{ s}$ to $t = 16\text{ s}$.
c) Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at $t = 16\text{ s}$.

Solution

a) For a third order polynomial, the velocity is given by

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$

Since we want to find the velocity at $t = 16$, and we are using a third order polynomial, we need to choose the four data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The four data points are $t_0 = 10,\ t_1 = 15,\ t_2 = 20,$ and $t_3 = 22.5$.

Then

$$t_0 = 10, \quad v(t_0) = 227.04$$

$$t_1 = 15, \quad v(t_1) = 362.78$$

$$t_2 = 20, \quad v(t_2) = 517.35$$

$$t_3 = 22.5, \quad v(t_3) = 602.97$$

gives

$$\begin{aligned} b_0 &= v[t_0] \\ &= v(t_0) \\ &= 227.04 \end{aligned}$$

$$\begin{aligned} b_1 &= v[t_1, t_0] \\ &= \frac{v(t_1) - v(t_0)}{t_1 - t_0} \\ &= \frac{362.78 - 227.04}{15 - 10} \\ &= 27.148 \end{aligned}$$

$$\begin{aligned} b_2 &= v[t_2, t_1, t_0] \\ &= \frac{v[t_2, t_1] - v[t_1, t_0]}{t_2 - t_0} \end{aligned}$$

$$\begin{aligned} v[t_2, t_1] &= \frac{v(t_2) - v(t_1)}{t_2 - t_1} \\ &= \frac{517.35 - 362.78}{20 - 15} \\ &= 30.914 \end{aligned}$$

$$v[t_1, t_0] = 27.148$$

$$b_2 = \frac{v[t_2, t_1] - v[t_1, t_0]}{t_2 - t_0}$$

$$= \frac{30.914 - 27.148}{20 - 10}$$
$$= 0.37660$$

$$b_3 = v[t_3, t_2, t_1, t_0]$$
$$= \frac{v[t_3, t_2, t_1] - v[t_2, t_1, t_0]}{t_3 - t_0}$$

$$v[t_3, t_2, t_1] = \frac{v[t_3, t_2] - v[t_2, t_1]}{t_3 - t_1}$$

$$v[t_3, t_2] = \frac{v(t_3) - v(t_2)}{t_3 - t_2}$$
$$= \frac{602.97 - 517.35}{22.5 - 20}$$
$$= 34.248$$

$$v[t_2, t_1] = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$
$$= \frac{517.35 - 362.78}{20 - 15}$$
$$= 30.914$$

$$v[t_3, t_2, t_1] = \frac{v[t_3, t_2] - v[t_2, t_1]}{t_3 - t_1}$$
$$= \frac{34.248 - 30.914}{22.5 - 15}$$
$$= 0.44453$$

$$v[t_2, t_1, t_0] = 0.37660$$

$$b_3 = \frac{v[t_3, t_2, t_1] - v[t_2, t_1, t_0]}{t_3 - t_0}$$
$$= \frac{0.44453 - 0.37660}{22.5 - 10}$$
$$= 5.4347 \times 10^{-3}$$

Hence

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$
$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15)$$
$$+ 5.5347 \times 10^{-3}(t - 10)(t - 15)(t - 20)$$

At $t = 16$,

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15)$$
$$+ 5.5347 \times 10^{-3}(16 - 10)(16 - 15)(16 - 20)$$
$$= 392.06 \text{ m/s}$$

b) The distance covered by the rocket between $t = 11\text{ s}$ and $t = 16\text{ s}$ can be calculated from the interpolating polynomial

$$v(t) = 227.04 + 27.148(t-10) + 0.37660(t-10)(t-15)$$
$$+ 5.5347\times10^{-3}(t-10)(t-15)(t-20)$$
$$= -4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3,\ 10 \le t \le 22.5$$

Note that the polynomial is valid between $t = 10$ and $t = 22.5$ and hence includes the limits of $t = 11$ and $t = 16$.

So

$$s(16) - s(11) = \int_{11}^{16} v(t)dt$$
$$= \int_{11}^{16} (-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3)dt$$
$$= \left[-4.2541t + 21.265\frac{t^2}{2} + 0.13204\frac{t^3}{3} + 0.0054347\frac{t^4}{4}\right]_{11}^{16}$$
$$= 1605 \text{ m}$$

c) The acceleration at $t = 16$ is given by

$$a(16) = \frac{d}{dt}v(t)\Big|_{t=16}$$
$$a(t) = \frac{d}{dt}v(t)$$
$$= \frac{d}{dt}\left(-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3\right)$$
$$= 21.265 + 0.26408t + 0.016304t^2$$
$$a(16) = 21.265 + 0.26408(16) + 0.016304(16)^2$$
$$= 29.664 \text{ m/s}^2$$

INTERPOLATION	
Topic	Newton's Divided Difference Interpolation
Summary	Textbook notes and examples of Newton's divided difference interpolation.
Major	Mechanical Engineering
Authors	Autar Kaw
Date	November 17, 2012
Web Site	http://numericalmethods.eng.usf.edu