# Chapter 08.02
# Euler's Method for Ordinary Differential Equations

*After reading this chapter, you should be able to*:

1. *develop Euler's Method for solving ordinary differential equations,*
2. *determine how the step size affects the accuracy of a solution,*
3. *derive Euler's formula from Taylor series, and*
4. *use Euler's method to find approximate values of integrals.*

### What is Euler's method?

Euler's method is a numerical technique to solve ordinary differential equations of the form

$$\frac{dy}{dx} = f(x,y), y(0) = y_0 \tag{1}$$

So only first order ordinary differential equations can be solved by using Euler's method. In another chapter we will discuss how Euler's method is used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations. How does one write a first order differential equation in the above form?

### Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x,y),\ \ y(0) = y_0 \text{ form.}$$

### Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x,y)=1.3e^{-x}-2y$$

**Example 2**

Rewrite

$$e^{y}\frac{dy}{dx}+x^{2}y^{2}=2\sin(3x),\ y(0)=5$$

in

$$\frac{dy}{dx}=f(x,y),\ y(0)=y_0 \text{ form.}$$

**Solution**

$$e^{y}\frac{dy}{dx}+x^{2}y^{2}=2\sin(3x),\ y(0)=5$$

$$\frac{dy}{dx}=\frac{2\sin(3x)-x^{2}y^{2}}{e^{y}},\ y(0)=5$$

In this case

$$f(x,y)=\frac{2\sin(3x)-x^{2}y^{2}}{e^{y}}$$

**Derivation of Euler's method**

At $x=0$, we are given the value of $y=y_0$. Let us call $x=0$ as $x_0$. Now since we know the slope of $y$ with respect to $x$, that is, $f(x,y)$, then at $x=x_0$, the slope is $f(x_0,y_0)$. Both $x_0$ and $y_0$ are known from the initial condition $y(x_0)=y_0$.

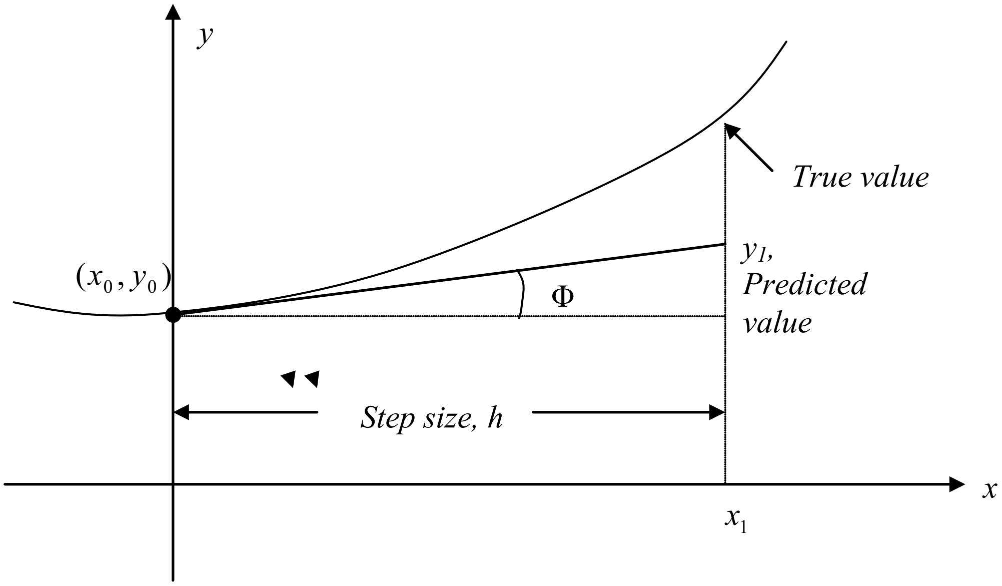

**Figure 1** Graphical interpretation of the first step of Euler's method.

So the slope at $x = x_0$ as shown in Figure 1 is

$$\begin{aligned}\text{Slope} &= \frac{Rise}{Run} \\ &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= f(x_0, y_0)\end{aligned}$$

From here

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

Calling $x_1 - x_0$ the step size $h$, we get

$$y_1 = y_0 + f(x_0, y_0)h \tag{2}$$

One can now use the value of $y_1$ (an approximate value of $y$ at $x = x_1$) to calculate $y_2$, and that would be the predicted value at $x_2$, given by

$$y_2 = y_1 + f(x_1, y_1)h$$
$$x_2 = x_1 + h$$

Based on the above equations, if we now know the value of $y = y_i$ at $x_i$, then

$$y_{i+1} = y_i + f(x_i, y_i)h \tag{3}$$

This formula is known as Euler's method and is illustrated graphically in Figure 2. In some books, it is also called the Euler-Cauchy method.

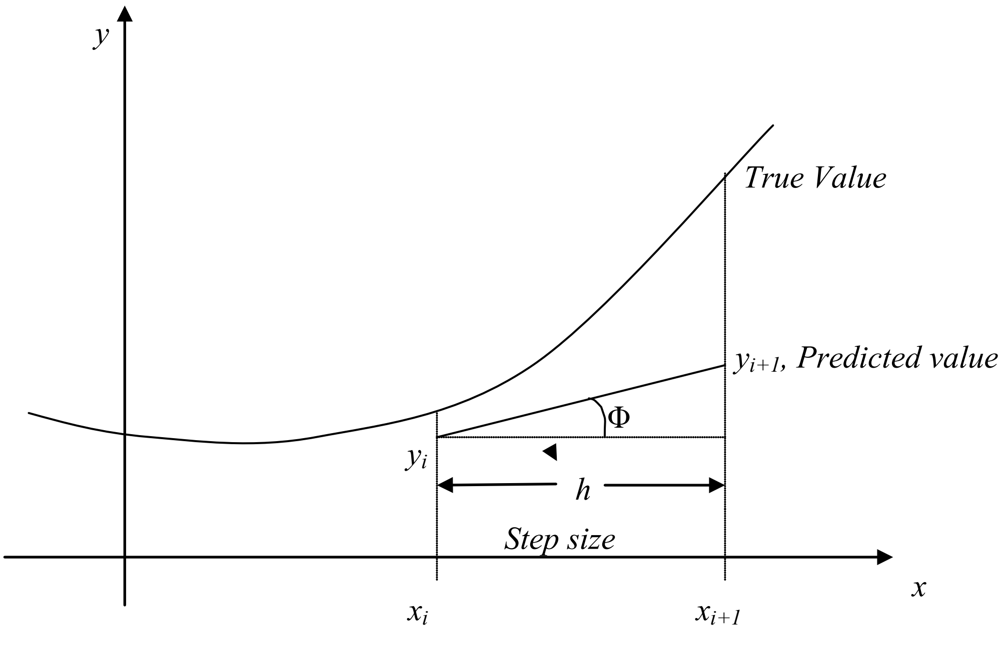

**Figure 2** General graphical interpretation of Euler's method.

**Example 3**

A solid steel shaft at room temperature of $27\ {}^{\circ}\mathrm{C}$ is needed to be contracted so that it can be shrunk-fit into a hollow hub. It is placed in a refrigerated chamber that is maintained at $-33\ {}^{\circ}\mathrm{C}$. The rate of change of temperature of the solid shaft $\theta$ is given by

$$\frac{d\theta}{dt} = -5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 \\ +5.42\times10^{-2}\theta + 5.588\end{pmatrix}(\theta+33)$$

$$\theta(0) = 27\ {}^{\circ}\mathrm{C}$$

Using Euler's method, find the temperature of the steel shaft after 86400 seconds. Take a step size of $h = 43200$ seconds.

**Solution**

$$\frac{d\theta}{dt} = -5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 \\ +5.42\times10^{-2}\theta + 5.588\end{pmatrix}(\theta+33)$$

$$f(t,\theta) = -5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 \\ +5.42\times10^{-2}\theta + 5.588\end{pmatrix}(\theta+33)$$

The Euler's method reduces to

$$\theta_{i+1} = \theta_i + f(t_i,\theta_i)h$$

For $i = 0$, $t_0 = 0$, $\theta_0 = 27$

$$\begin{aligned}\theta_1 &= \theta_0 + f(t_0,\theta_0)h \\ &= 27 + f(0,27)43200 \\ &= 27 + \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(27)^4 + 2.33\times10^{-5}(27)^3 \\ +1.35\times10^{-3}(27)^2 + 5.42\times10^{-2}(27) + 5.588\end{pmatrix}(27+33)\right)43200 \\ &= 27 + (-0.0020893)43200 \\ &= -63.258\ {}^{\circ}\mathrm{C}\end{aligned}$$

$\theta_1$ is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 43200 = 43200\ \mathrm{s}$$

$$\theta(43200) \approx \theta_1 = -63.258\ {}^{\circ}\mathrm{C}$$

For $i = 1$, $t_1 = 43200$, $\theta_1 = -63.258$

$$\begin{aligned}\theta_2 &= \theta_1 + f(t_1,\theta_1)h \\ &= -63.258 + f(43200,-63.258)43200 \\ &= -63.258 + \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-63.258)^4 + 2.33\times10^{-5}(-63.258)^3 \\ +1.35\times10^{-3}(-63.258)^2 \\ +5.42\times10^{-2}(-63.258) + 5.588\end{pmatrix}(-63.258+33)\right)43200 \\ &= -63.258 + (-0.0092607)43200 \\ &= -463.32\ {}^{\circ}\mathrm{C}\end{aligned}$$

$\theta_2$ is the approximate temperature at

$$t = t_2 = t_1 + h = 43200 + 43200 = 86400 \text{ s}$$

$$\theta(86400) \approx \theta_2 = -463.32\,^{\circ}\text{C}$$

Figure 3 compares the exact solution with the numerical solution from Euler's method for the step size of $h = 43200$.

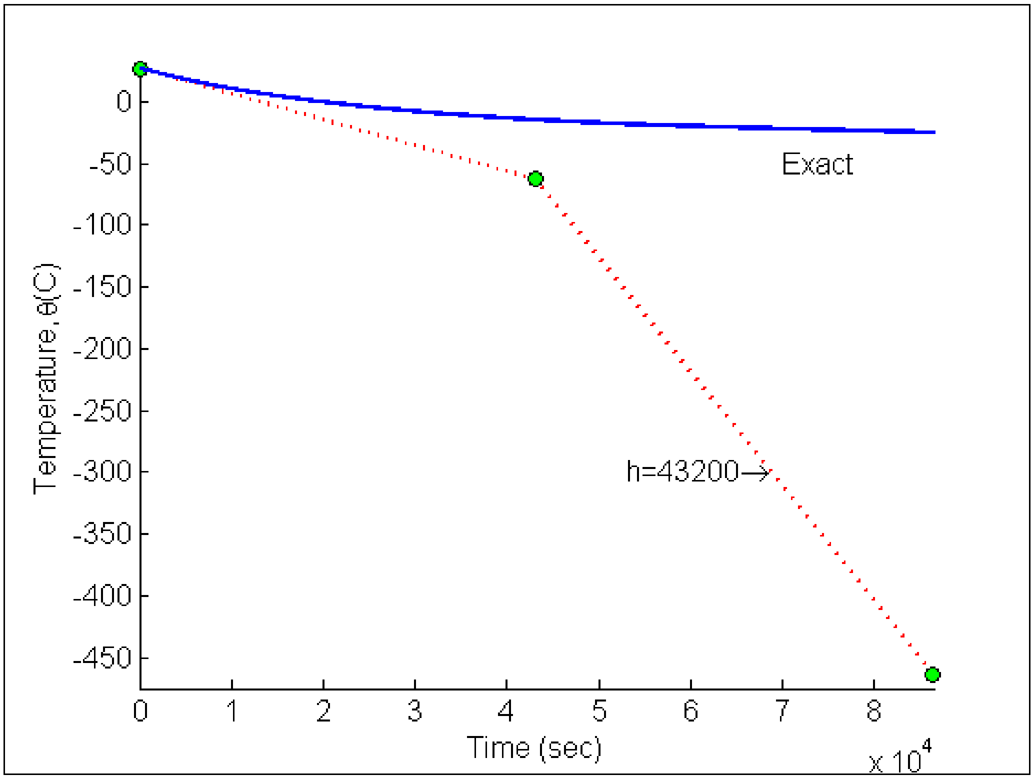

**Figure 3** Comparing exact and Euler's method.

The problem was solved again using smaller step sizes. The results are given below in Table 1.

**Table 1** Temperature at 86400 seconds as a function of step size, $h$.

| Step size, $h$ | $\theta(86400)$ | $E_t$ | $\lvert\epsilon_t\rvert$ % |
|---|---|---|---|
| 86400 | −153.52 | 127.42 | 488.21 |
| 43200 | −463.32 | 437.22 | 1675.2 |
| 21600 | −29.542 | 3.4421 | 13.189 |
| 10800 | −27.795 | 1.6962 | 6.4988 |
| 5400 | −26.958 | 0.85870 | 3.2902 |

Figure 4 shows how the temperature varies as a function of time for different step sizes.

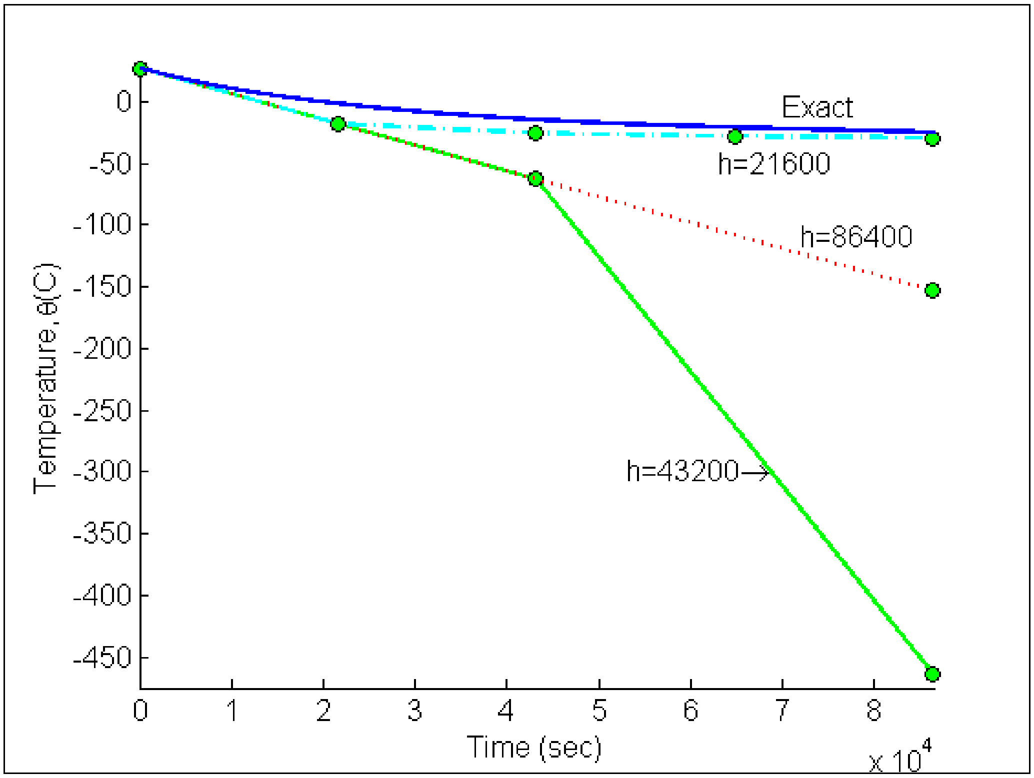

**Figure 4** Comparison of Euler's method with exact solution for different step sizes.

While the values of the calculated temperature at $t = 86400\,\text{s}$ as a function of step size are plotted in Figure 5.

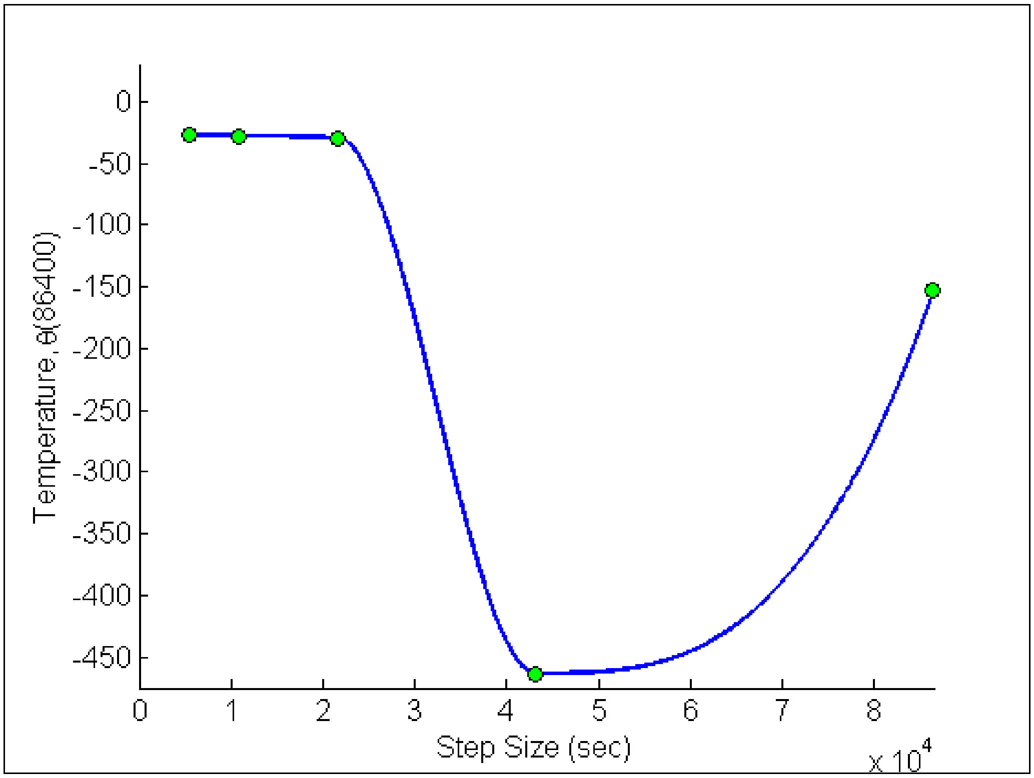

**Figure 5** Effect of step size in Euler's method.

The solution to this nonlinear equation at $t = 86400\,\text{s}$ is

$$\theta(86400) = -26.099^\circ\text{C}$$

**Can one solve a definite integral using numerical methods such as Euler's method of solving ordinary differential equations?**

Let us suppose you want to find the integral of a function $f(x)$

$$I = \int_a^b f(x)dx\,.$$

Both fundamental theorems of calculus would be used to set up the problem so as to solve it as an ordinary differential equation.

The first fundamental theorem of calculus states that if $f$ is a continuous function in the interval [a,b], and $F$ is the antiderivative of $f$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

The second fundamental theorem of calculus states that if $f$ is a continuous function in the open interval $D$, and $a$ is a point in the interval $D$, and if

$$F(x) = \int_a^x f(t)dt$$

then

$$F'(x) = f(x)$$

at each point in $D$.

Asked to find $\int_a^b f(x)dx$, we can rewrite the integral as the solution of an ordinary differential equation (here is where we are using the second fundamental theorem of calculus)

$$\frac{dy}{dx} = f(x),\ y(a) = 0,$$

where then $y(b)$ (here is where we are using the first fundamental theorem of calculus) will give the value of the integral $\int_a^b f(x)dx$.

**Example 4**

Find an approximate value of

$$\int_5^8 6x^3 dx$$

using Euler's method of solving an ordinary differential equation. Use a step size of $h = 1.5$.

**Solution**

Given $\int_5^8 6x^3 dx$, we can rewrite the integral as the solution of an ordinary differential equation

$$\frac{dy}{dx} = 6x^3,\ y(5) = 0$$

where $y(8)$ will give the value of the integral $\int_5^8 6x^3 dx$.

$$\frac{dy}{dx} = 6x^3 = f(x,y),\ y(5) = 0$$

The Euler's method equation is

$$y_{i+1} = y_i + f(x_i, y_i)h$$

<u>Step 1</u>

$$i = 0,\ x_0 = 5,\ y_0 = 0$$

$$h = 1.5$$

$$\begin{aligned} x_1 &= x_0 + h \\ &= 5 + 1.5 \\ &= 6.5 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)h \\ &= 0 + f(5,0) \times 1.5 \\ &= 0 + \left(6 \times 5^3\right) \times 1.5 \end{aligned}$$

$$
\begin{aligned}
&= 1125 \\
&\approx y(6.5)
\end{aligned}
$$

<u>Step 2</u>

$$i = 1,\ x_1 = 6.5,\ y_1 = 1125$$

$$
\begin{aligned}
x_2 &= x_1 + h \\
&= 6.5 + 1.5 \\
&= 8 \\
y_2 &= y_1 + f(x_1, y_1)h \\
&= 1125 + f(6.5, 1125) \times 1.5 \\
&= 1125 + \left(6 \times 6.5^3\right) \times 1.5 \\
&= 3596.625 \\
&\approx y(8)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_5^8 6x^3 dx &= y(8) - y(5) \\
&\approx 3596.625 - 0 \\
&= 3596.625
\end{aligned}
$$

| ORDINARY DIFFERENTIAL EQUATIONS | |
|---|---|
| Topic | Euler's Method for ordinary differential equations |
| Summary | Textbook notes on Euler's method for solving ordinary differential equations |
| Major | Mechanical Engineering |
| Authors | Autar Kaw |
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