Holistic Numerical Methods Institute

committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Gauss-Seidel Method of Solving Simultaneous Linear Equations
COMPLETE SOLUTION SET

1. A square matrix $[A]_{n\times n}$ is diagonally dominant if

(A) $|a_{ii}| \geq \sum_{\substack{j=1\\ i\neq j}}^{n} |a_{ij}|,\ i=1,2,...,n$

(B) $|a_{ii}| \geq \sum_{\substack{j=1\\ i\neq j}}^{n} |a_{ij}|,\ i=1,2,...,n$ and $|a_{ii}| > \sum_{\substack{j=1\\ i\neq j}}^{n} |a_{ij}|,$ for any $i=1,2,...,n$

(C) $|a_{ii}| \geq \sum_{j=1}^{n} |a_{ij}|,\ i=1,2,...,n$ and $|a_{ii}| > \sum_{j=1}^{n} |a_{ij}|,$ for any $i=1,2,...,n$

(D) $|a_{ii}| \geq \sum_{j=1}^{n} |a_{ij}|,\ i=1,2,...,n$

Solution

The correct answer is (B).

A $n\times n$ square matrix $[A]$ is a diagonally dominant matrix if $|a_{ii}| \geq \sum_{\substack{j=1\\ i\neq j}}^{n} |a_{ij}|,$ for all $i = 1, 2, \ldots, n$

and $|a_{ii}| > \sum_{\substack{j=1\\ i\neq j}}^{n} |a_{ij}|$ for at least one i, that is, for each row, the absolute value of the diagonal element is greater than or equal to the sum of the absolute values of the rest of the elements of that row, and that the inequality is strictly greater than for at least one row. Diagonally dominant matrices are important in ensuring convergence in iterative schemes of solving simultaneous linear equations.

$$[A] = \begin{bmatrix} 15 & 6 & 7 \\ 2 & -4 & -2 \\ 3 & 2 & 6 \end{bmatrix}$$ is a diagonally dominant matrix as

$$|a_{11}| = |15| = 15 \geq |a_{12}| + |a_{13}| = |6| + |7| = 13$$
$$|a_{22}| = |-4| = 4 \geq |a_{21}| + |a_{23}| = |2| + |-2| = 4$$
$$|a_{33}| = |6| = 6 \geq |a_{31}| + |a_{32}| = |3| + |2| = 5$$

and for at least one row, that is Rows 1 and 3 in this case, the inequality is a strictly greater than inequality.

2. Using $[x_1, x_2, x_3] = [1,3,5]$ as the initial guess, the values of $[x_1, x_2, x_3]$ after three iterations in the Gauss-Seidel method for

$$\begin{bmatrix} 12 & 7 & 3 \\ 1 & 5 & 1 \\ 2 & 7 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}$$

are

(A) [-2.8333 -1.4333 -1.9727]
(B) [1.4959 -0.90464 -0.84914]
(C) [0.90666 -1.0115 -1.0243]
(D) [1.2148 -0.72060 -0.82451]

Solution
The correct answer is (C).

Rewriting the equations gives

$$x_1 = \frac{2 - 7x_2 - 3x_3}{12}$$

$$x_2 = \frac{-5 - x_1 - x_3}{5}$$

$$x_3 = \frac{6 - 2x_1 - 7x_2}{-11}$$

Iteration #1

Given the initial guess of the solution vector as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

we get

$$x_1 = \frac{2 - 7(3) - 3(5)}{12}$$
$$= -2.8333$$

$$x_2 = \frac{-5 - (-2.8333) - (5)}{5}$$
$$= -1.4333$$

$$x_3 = \frac{6 - 2(-2.8333) - 7(-1.4333)}{-11}$$
$$= -1.9727$$

At the end of the first iteration, the estimate of the solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2.8333 \\ -1.4333 \\ -1.9727 \end{bmatrix}$$

Iteration #2

Since the estimate of the solution vector at the end of Iteration #1 is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2.8333 \\ -1.4333 \\ -1.9727 \end{bmatrix}$$

we get

$$\begin{aligned} x_1 &= \frac{2 - 7(-1.4333) - 3(-1.9727)}{12} \\ &= 1.4960 \\ x_2 &= \frac{-5 - (1.4960) - (-1.9727)}{5} \\ &= -0.90465 \\ x_3 &= \frac{6 - 2(1.4960) - 7(-0.90464)}{-11} \\ &= -0.84915 \end{aligned}$$

At the end of the second iteration, the estimate of the solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.4960 \\ -0.90465 \\ -0.84915 \end{bmatrix}$$

Iteration #3

Since the estimate of the solution vector at the end of Iteration #2 is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.4960 \\ -0.90465 \\ -0.84915 \end{bmatrix}$$

we get

$$x_1 = \frac{2-7(-0.90465)-3(-0.84915)}{12}$$
$$= 0.90666$$
$$x_2 = \frac{-5-(0.90666)-(-0.84914)}{5}$$
$$= -1.0115$$
$$x_3 = \frac{6-2(0.90666)-7(-1.0115)}{-11}$$
$$= -1.0243$$

At the end of the third iteration, the estimate of the solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.90666 \\ -1.0115 \\ -1.0243 \end{bmatrix}$$

3. To ensure that the following system of equations,

$$\begin{aligned} 2x_1 + 7x_2 - 11x_3 &= 6 \\ x_1 + 2x_2 + x_3 &= -5 \\ 7x_1 + 5x_2 + 2x_3 &= 17 \end{aligned}$$

converges using the Gauss-Seidel method, one can rewrite the above equations as follows:

(A) $$\begin{bmatrix} 2 & 7 & -11 \\ 1 & 2 & 1 \\ 7 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 17 \end{bmatrix}$$

(B) $$\begin{bmatrix} 7 & 5 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17 \\ -5 \\ 6 \end{bmatrix}$$

(C) $$\begin{bmatrix} 7 & 5 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 17 \end{bmatrix}$$

(D) The equations cannot be rewritten in a form to ensure convergence.

Solution

The correct answer is (B).

A system of equations will converge using the Gauss-Seidel method if the coefficient matrix is diagonally dominant. Thus, rewriting the system of equations as

$$\begin{bmatrix} 7 & 5 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17 \\ -5 \\ 6 \end{bmatrix}$$

results in the coefficient matrix being diagonally dominant.

$$\begin{aligned} |a_{11}| &= |7| = 7 \geq |a_{12}| + |a_{13}| = |5| + |2| = 7 \\ |a_{22}| &= |2| = 2 \geq |a_{21}| + |a_{23}| = |1| + |1| = 2 \\ |a_{33}| &= |-11| = 11 \geq |a_{31}| + |a_{32}| = |2| + |7| = 9 \end{aligned}$$

4. For $\begin{bmatrix} 12 & 7 & 3 \\ 1 & 5 & 1 \\ 2 & 7 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \\ -2 \end{bmatrix}$ and using $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$ as the initial guess, the values of $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$ are found at the end of each iteration as

Iteration #	x_1	x_2	x_3
1	0.41667	1.1167	0.96818
2	0.93990	1.0184	1.0008
3	0.98908	1.0020	0.99931
4	0.99899	1.0003	1.0000

At what first iteration number would you trust at least 1 significant digit in your solution?
(A) 1
(B) 2
(C) 3
(D) 4

Solution
The correct answer is (C).

The absolute relative approximate error at the end of the first iteration is

$$\left|\in_a\right|_1 = \left|\frac{0.41667 - 1.0000}{0.41667}\right| \times 100$$

$$= 140.00\%$$

$$\left|\in_a\right|_2 = \left|\frac{1.1167 - 2.0000}{1.1167}\right| \times 100$$

$$= 79.104\%$$

$$\left|\in_a\right|_3 = \left|\frac{0.96818 - 1.0000}{0.96818}\right| \times 100$$

$$= 3.2864\%$$

The maximum absolute relative approximate error is 140%.
Since, $140\% \le 0.5 \times 10^{2-m}$, $m = -1$. There are no significant digits correct. We are looking for 1 significant digit, so we must continue to conduct iterations.

The absolute relative approximate error at the end of the second iteration is

$$\left|\in_a\right|_1 = \left|\frac{0.93990 - 0.41667}{0.93990}\right| \times 100$$

$$= 55.669\%$$

$$\left|\in_a\right|_2 = \left|\frac{1.0184 - 1.1167}{1.0184}\right| \times 100$$

$$= 9.6509\%$$

$$\left|\in_a\right|_3 = \left|\frac{1.0008 - 0.96818}{1.0008}\right| \times 100$$

$$= 3.2564\%$$

The maximum absolute relative approximate error is 55.669%.
Since, $55.669\% \leq 0.5 \times 10^{2-m}$, $m = -1$. There are no significant digits correct. We are looking for 1 significant digit so we must continue to conduct more iterations.

The absolute relative approximate error at the end of the third iteration is

$$\left|\in_a\right|_1 = \left|\frac{0.98908 - 0.9399}{0.98908}\right| \times 100$$

$$= 4.9727\%$$

$$\left|\in_a\right|_2 = \left|\frac{1.0020 - 1.0184}{1.0020}\right| \times 100$$

$$= 1.6322\%$$

$$\left|\in_a\right|_3 = \left|\frac{0.99930 - 1.0007}{0.99930}\right| \times 100$$

$$= 0.14660\%$$

The maximum absolute approximate error is 4.9733%.
Since, $4.9733\% \leq 0.5 \times 10^{2-m}$, $m = 1$. There is at least one significant digit correct. Since we were looking for 1 correct significant digit, we need not conduct any more iterations.

5. The algorithm for the Gauss-Seidel method to solve $[A][X]=[C]$ is given as follows when using $n\max$ iterations. The initial value of $[X]$ is stored in $[X]$.

(A)
```
Sub Seidel(n, a, x, rhs, nmax)
For k = 1 To nmax
For i = 1 To n
For j = 1 To n
If (i <> j) Then
Sum = Sum + a(i, j) * x(j)
endif
Next j
x(i) = (rhs(i) - Sum) / a(i,i)
Next i
Next j
End Sub
```

(B)
```
Sub Seidel(n, a, x, rhs, nmax)
For k = 1 To nmax
  For i = 1 To n
  Sum = 0
  For j = 1 To n
  If (i <> j) Then
  Sum = Sum + a(i, j) * x(j)
  endif
  Next j
  x(i) = (rhs(i) - Sum) / a(i,i)
  Next i
  Next k
  End Sub
```

(C)
```
Sub Seidel(n, a, x, rhs, nmax)
For k = 1 To nmax
For i = 1 To n
Sum = 0
For j = 1 To n
Sum = Sum + a(i, j) * x(j)
Next j
x(i) = (rhs(i) - Sum) / a(i,i)
Next i
Next k
End Sub
```

(D) Sub Seidel $(n, a, x, rhs, \text{nmax})$

```
For k = 1 To nmax
For i = 1 To n
Sum = 0
For j = 1 To n
If (i <> j) Then
Sum = Sum + a(i, j) * x(j)
endif
Next j
x(i) = (rhs(i) – Sum) / a(i,i)
Next i
Next k
End Sub
```

Solution

The correct answer is (B).

```
Sub Seidel(n, a, x, rhs, nmax)
    For k = 1 To nmax
    For i = 1 To n
    Sum = 0
    For j = 1 To n
    If (i <> j) Then
    Sum = Sum + a(i, j) * x(j)
    endif

    Next j
    x(i) = (rhs(i) - Sum) / a(i, i)
    Next i
    Next k
    End Sub
```

Choice (A) is incorrect because the value of the variable ‘Sum’ needs to be reset on each row. Choice (C) is incorrect because it does not include an if statement that only adds the sum when the value of ‘i’ is not equal to the value of ‘j’. The value of ‘Sum’ would include the value of ‘a(i,i)*x(i)’, which would give an incorrect answer for the value of ‘x(i)’. Choice (D) is incorrect because it does not subtract the value of ‘Sum’ from ‘rhs(i)’, which again, would give the incorrect answer for ‘x(i)’.

6. Thermistors measure temperature, have a nonlinear output and are valued for a limited range. So when a thermistor is manufactured, the manufacturer supplies a resistance vs. temperature curve. An accurate representation of the curve is generally given by

$$\frac{1}{T} = a_0 + a_1 \ln(R) + a_2 \{\ln(R)\}^2 + a_3 \{\ln(R)\}^3$$

where T is temperature in Kelvin, R is resistance in ohms, and a_0, a_1, a_2, a_3 are constants of the calibration curve. Given the following for a thermistor

R	T
ohm	°C
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128

the value of temperature in °C for a measured resistance of 900 ohms most nearly is

(A) 30.002
(B) 30.473
(C) 31.272
(D) 31.445

Solution

The correct answer is (B).

Given

$$\frac{1}{T} = a_0 + a_1 \ln(R) + a_2 \{\ln(R)\}^2 + a_3 \{\ln(R)\}^3$$

Then from the data in the table we can write four equations

$$\frac{1}{25.113} = a_0 + a_1 \ln(1101) + a_2 \{\ln(1101)\}^2 + a_3 \{\ln(1101)\}^3$$
$$\frac{1}{30.131} = a_0 + a_1 \ln(911.3) + a_2 \{\ln(911.3)\}^2 + a_3 \{\ln(911.3)\}^3$$
$$\frac{1}{40.120} = a_0 + a_1 \ln(636) + a_2 \{\ln(636)\}^2 + a_3 \{\ln(636)\}^3$$
$$\frac{1}{50.128} = a_0 + a_1 \ln(451.1) + a_2 \{\ln(451.1)\}^2 + a_3 \{\ln(451.1)\}^3$$

which reduce to

$$0.039820 = a_0 + 7.0040 a_1 + 49.056 a_2 + 343.58 a_3$$
$$0.033188 = a_0 + 6.8149 a_1 + 46.442 a_2 + 316.50 a_3$$
$$0.024925 = a_0 + 6.4552 a_1 + 41.670 a_2 + 268.99 a_3$$
$$0.019949 = a_0 + 6.1117 a_1 + 37.353 a_2 + 228.29 a_3$$

In matrix form the equations can be rewritten as

$$\begin{bmatrix} 343.58 & 49.056 & 7.0040 & 1 \\ 316.5 & 46.442 & 6.8149 & 1 \\ 268.99 & 41.670 & 6.4552 & 1 \\ 228.29 & 37.353 & 6.1117 & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0.039820 \\ 0.03318 \\ 0.024925 \\ 0.019949 \end{bmatrix}$$

Using any method gives

$$a_3 = 0.011173$$

$$a_2 = -0.20448$$

$$a_1 = 1.2605$$

$$a_0 = -2.5964$$

$$\frac{1}{T} = -2.5964 + 1.2605 \ln(R) + -0.20448\{\ln(R)\}^2 + 0.011173\{\ln(R)\}^3$$

$$= -2.5964 + 1.2605 \ln(900) + -0.20448\{\ln(900)\}^2 + 0.011173\{\ln(900)\}^3$$

$$= 0.032816$$

$$T = \frac{1}{0.032816}$$

$$= 30.473\,^\circ\text{C}$$