Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Background
Interpolation
COMPLETE SOLUTION SET

1. The number of different polynomials that can go through two fixed data points (x_1, y_1) and (x_2, y_2) is
   A) 0
   B) 1
   C) 2
   D) infinite

Solution
The correct answer is (D).

The number of polynomials that can go through two fixed data points (x_1, y_1) and (x_2, y_2) is infinite. For example the polynomials $y = 5x - 6$ and $y = x^2$ go through the two data points (2,4) and (3,9). So does every polynomial of the form $y = ax^m + bx^n$, where m and n are any positive integers of your choice.

2. Given n+1 data pairs, a unique polynomial of degree __________________ passes through $n+1$ data points.

(A) $n+1$
(B) $n+1$ or less
(C) n
(D) n or less

Solution
The correct answer is (D).

A unique polynomial of degree n or less passes through $n+1$ data points. If the polynomial is not unique, then at least two polynomials of order n or less pass through the $n+1$ data points. Assume two polynomials $P_n(x)$ and $Q_n(x)$ go through $n+1$ data points,

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$$

Then

$$R_n(x) = P_n(x) - Q_n(x)$$

Since $P_n(x)$ and $Q_n(x)$ pass through all the $n+1$ data points,

$$P_n(x_i) = Q_n(x_i), \; i = 0, \ldots, n$$

Hence

$$R_n(x_i) = P_n(x_i) - Q_n(x_i) = 0, \; i = 0, \ldots, n$$

The n^{th} order polynomial $R_n(x)$ has $n+1$ zeros. A polynomial of order n can have $n+1$ zeros only if it is identical to a zero polynomial, that is,

$$R_n(x) \equiv 0$$

Hence

$$P_n(x) \equiv Q_n(x)$$

Extra Notes for the Student:
How can one show that if a second order polynomial has three zeros, then it is zero everywhere? If $R_2(x) = a_0 + a_1 x + a_2 x^2$, then if it has three zeros at x_1, x_2, and x_3, then

$$R_2(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 = 0$$
$$R_2(x_2) = a_0 + a_1 x_2 + a_2 x_2^2 = 0$$
$$R_2(x_3) = a_0 + a_1 x_3 + a_2 x_3^2 = 0$$

which in matrix form gives

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The above set of equations has a trivial solution, that is, $a_1 = a_2 = a_3 = 0$. But is this the only solution? That is true if the coefficient matrix is invertible.

The determinant of the coefficient matrix can be found symbolically with the forward elimination steps of naïve Gauss elimination to give

$$\det\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = x_2x_3^2 - x_2^2x_3 - x_1x_3^2 + x_1^2x_3 + x_1x_2^2 - x_1^2x_2$$

$$= (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$$

Since

$$x_1 \neq x_2 \neq x_3$$

the determinant is non-zero. Hence the coefficient matrix is invertible. $a_1 = a_2 = a_3 = 0$ is the only solution, that is, $R_2(x) \equiv 0$.

3. The following function(s) can be used for interpolation:

(A) polynomial
(B) exponential
(C) trigonometric
(D) all of the above

Solution

The correct answer is (D).

Polynomials are often used for interpolation because they are easy to evaluate, differentiate and integrate. However, other functions such as trigonometric and exponential can be used for interpolation. How is a polynomial easy to evaluate as compared to a trigonometric function? Because terms such as x^m in a polynomial involve multiplication of x to itself $m-1$ times. However, trigonometric and exponential functions include the use of computationally more involved calculations via a Taylor series.

4. Polynomials are the most commonly used functions for interpolation because they are easy to

(A) evaluate
(B) differentiate
(C) integrate
(D) evaluate, differentiate and integrate

Solution

The correct answer is (D).

Polynomials are often used for interpolation because they are easy to evaluate, differentiate and integrate. However, other functions such as trigonometric and exponential can be used for interpolation. How is a polynomial easy to evaluate as compared to a trigonometric function? Because terms such as x^m in a polynomial involve multiplication of x to itself $m-1$ times. However, trigonometric and exponential functions include the use of computationally more involved calculations via a Taylor series.

5. Given $n+1$ data points $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, assume you pass a function $f(x)$ through all the data points. If now the value of the function $f(x)$ is required to be found outside the range of the given x-data, the procedure is called

(A) extrapolation
(B) interpolation
(C) guessing
(D) regression

Solution

The correct answer is (A).

If x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

6. Given three data points (1,6), (3,28), and (10, 231), it is found that the function $y = 2x^2 + 3x + 1$ passes through the three data points. Your estimate of y at $x = 2$ is most nearly

(A) 6
(B) 15
(C) 17
(D) 28

Solution

The correct answer is (B).

$$
\begin{aligned}
y(x) &= 2x^2 + 3x + 1 \\
y(2) &= 2 \times 2^2 + 3 \times 2 + 1 \\
&= 8 + 6 + 1 \\
&= 15
\end{aligned}
$$