Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Gauss Quadrature Rule
Integration
COMPLETE SOLUTION SET

1. $\int_{5}^{10} f(x)dx$ is exactly

(A) $\int_{-1}^{1} f(2.5x+7.5)dx$

(B) $2.5\int_{-1}^{1} f(2.5x+7.5)dx$

(C) $5\int_{-1}^{1} f(5x+5)dx$

(D) $5\int_{-1}^{1} (2.5x+7.5)f(x)dx$

Solution

The correct answer is (B).

$$\int_{a}^{b} f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}x+\frac{b+a}{2}\right)\frac{b-a}{2}dx$$

where

$a=5$

$b=10$

Thus

$$\begin{aligned}\int_{5}^{10} f(x)dx &= \int_{-1}^{1} f\left(\frac{10-5}{2}x+\frac{10+5}{2}\right)\frac{10-5}{2}dx \\ &= \int_{-1}^{1} f(2.5x+7.5)2.5dx \\ &= 2.5\int_{-1}^{1} f(2.5x+7.5)dx\end{aligned}$$

2. For a definite integral of any third order polynomial, the two-point Gauss quadrature rule will give the same results as the
   (A) 1-segment trapezoidal rule
   (B) 2-segment trapezoidal rule
   (C) 3-segment trapezoidal rule
   (D) Simpson's 1/3 rule

Solution

The correct answer is (D).

For integrating any third order polynomial, the two-point Gauss quadrature rule will give the same results as Simpson's 1/3 rule. Both these rules exactly integrate polynomials of third order or less.

3. The value of $\int_{0.2}^{2.2} xe^{x}dx$ by using the two-point Gauss quadrature rule is most nearly

(A) 11.672
(B) 11.807
(C) 12.811
(D) 14.633

Solution

The correct answer is (A).

First, change the limits of integration from [0.2, 2.2] to [-1, 1].

$$\int_{0.2}^{2.2} f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}x+\frac{b+a}{2}\right)\frac{b-a}{2}dx$$

$$= \int_{-1}^{1} f\left(\frac{2.2-0.2}{2}x+\frac{2.2+0.2}{2}\right)\frac{2.2-0.2}{2}dx$$

$$= 1\int_{-1}^{1} f(x+1.2)dx$$

Next, get weighting factors and function argument values for the two point rule

$$c_1 = 1.000000000$$

$$x_1 = -0.577350269$$

$$c_2 = 1.000000000$$

$$x_2 = 0.577350269$$

Now we can use the Gauss quadrature formula.

$$\int_{-1}^{1} f(x+1.2)dx \approx \left[c_1 f(x_1+1.2)+c_2 f(x_2+1.2)\right]$$

$$= \left[1\times f(-0.5773503+1.2)+1\times f(0.5773503+1.2)\right]$$

$$= \left[f(0.62265)+f(1.7774)\right]$$

$$= \left[\left(0.62265e^{0.62265}\right)+\left(1.7774e^{1.7774}\right)\right]$$

$$= \left[(1.1605)+(10.512)\right]$$

$$= 11.672$$

4. A scientist uses the one-point Gauss quadrature rule based on getting exact results of integration for functions $f(x)=1$ and x. The one-point Gauss quadrature rule approximation for $\int_a^b f(x)dx$ is

(A) $\frac{b-a}{2}[f(a)+f(b)]$

(B) $(b-a)f\left(\frac{a+b}{2}\right)$

(C) $\frac{b-a}{2}\left[f\left(\frac{b-a}{2}\left\{-\frac{1}{\sqrt{3}}\right\}+\frac{b+a}{2}\right)+f\left(\frac{b-a}{2}\left\{\frac{1}{\sqrt{3}}\right\}+\frac{b+a}{2}\right)\right]$

(D) $(b-a)f(a)$

Solution

The correct answer is (B).

The one-point Gauss quadrature rule is

$$\int_a^b f(x)dx \approx c_1 f(x_1)$$

The values of c_1 and x_1 are found by assuming the formula gives exact values for $\int_{-1}^{1} 1dx$ and $\int_{-1}^{1} xdx$.

$$\int_a^b 1dx = b-a = c_1$$

$$\int_a^b xdx = \frac{b^2-a^2}{2} = c_1 x_1$$

Since $c_1 = b-a,$ the other equation becomes

$$(b-a)x_1 = \frac{b^2-a^2}{2}$$

$$= \frac{(b-a)(b+a)}{2}$$

$$x_1 = \frac{b+a}{2}$$

Therefore, the one-point Gauss quadrature rule can be expressed as

$$\int_a^b f(x)dx \approx c_1 f(x_1)$$

$$= (b-a) f\left(\frac{b+a}{2}\right)$$

Note to student: Repeat the exercise by assuming that the formula gives exact results for integrals of the form

$$\int_a^b (a_0 + a_1 x)dx$$

5. A scientist develops an approximate formula for integration as

$$\int_a^b f(x)dx \approx c_1 f(x_1), \text{ where } a \le x_1 \le b$$

The values of c_1 and x_1 are found by assuming that the formula is exact for functions of the form $a_0 x + a_1 x^2$. The resulting formula would therefore be exact for integrating

(A) $f(x) = 2$
(B) $f(x) = 2 + 3x + 5x^2$
(C) $f(x) = 5x^2$
(D) $f(x) = 2 + 3x$

Solution

The correct answer is (C).

The only integrand that follows the form $a_0 x + a_1 x^2$ is $f(x) = 5x^2$ where

$a_0 = 0$
$a_1 = 5$

Choice (A) is incorrect because it contains a constant term, that is, 2. Choice (B) is incorrect as it contains a constant term, that is, 2. Choice (D) is incorrect as it contains a constant term, that is, 2. The form $a_0 x + a_1 x^2$ does not contain a constant term. It is a special case of a 2nd order polynomial with no constant term.

6. You are asked to estimate the water flow rate in a pipe of radius 2 m at a remote area location with a harsh environment. You already know that velocity varies along the radial location, but you do not know how it varies. The flow rate Q is given by

$$Q = \int_0^2 2\pi r V dr$$

To save money, you are allowed to put only two velocity probes (these probes send the data to the central office in New York, NY via satellite) in the pipe. Radial location, r is measured from the center of the pipe, that is $r = 0$ is the center of the pipe and $r = 2\text{m}$ is the pipe radius. The radial locations you would suggest for the two velocity probes for the most accurate calculation of the flow rate are

(A) 0, 2
(B) 1, 2
(C) 0, 1
(D) 0.42, 1.58

Solution
The correct answer is (D).

$$Q = \int_0^2 2\pi r V dr$$

$$= \int_0^2 f(r) dr$$

where

$$f(r) = 2\pi r V$$

Convert the $Q = \int_0^2 f(r) dr$ integral into an integral with limits [-1,1].

$$\int_0^2 f(r)dr = \int_{-1}^1 f\left(\frac{b-a}{2}r + \frac{b+a}{2}\right)\frac{b-a}{2}dr$$

$$= \int_{-1}^1 f\left(\frac{2-0}{2}r + \frac{2+0}{2}\right)\frac{2-0}{2}dr$$

$$= 1\int_{-1}^1 f(r+1)dr$$

The weighting factors and function argument values for the two point rule are

$$c_1 = 1.000000000$$

$$r_1 = -0.577350269$$

$$c_2 = 1.000000000$$

$$r_2 = 0.577350269$$

The Gauss quadrature formula is therefore

$$\int_{-1}^{1} f(r+1)dr \approx \left[c_1 f(r_1+1) + c_2 f(r_2+1)\right]$$

$$= \left[f(-0.5773503+1) + f(0.5773503+1)\right]$$

$$= \left[f(0.42) + f(1.58)\right]$$

By taking measurements of the velocity at 0.42 m and 1.58 m the flow rate can be approximated with up to third order accuracy.