Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Romberg Rule
Integration
COMPLETE SOLUTION SET

1. If I_n is the value of $\int_a^b f(x)dx$ using the n-segment trapezoidal rule, a better estimate of the integral can be found using Richardson's extrapolation as

(A) $I_{2n} + \frac{I_{2n} - I_n}{15}$

(B) $I_{2n} + \frac{I_{2n} - I_n}{3}$

(C) I_{2n}

(D) $I_{2n} + \frac{I_{2n} - I_n}{I_{2n}}$

Solution

The correct answer is (B).

Error in Multiple-Segment Trapezoidal Rule

The true error obtained when using the multiple segment trapezoidal rule with n segments to approximate an integral

$$I = \int_a^b f(x)dx \qquad (1)$$

is given by

$$E_t = -\frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\xi_i)}{n} \qquad (2)$$

where for each i, ξ_i is a point somewhere in the domain $[a+(i-1)h, a+ih]$, and the term $\frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$ can be viewed as an approximate average value of $f''(x)$ in $[a,b]$. This leads us to say that the true error E_t in Equation (2) is approximately proportional to

$$E_t \approx \alpha \frac{1}{n^2} \qquad (3)$$

for the estimate of $\int_a^b f(x)dx$ using the n-segment trapezoidal rule.

Richardson's Extrapolation Formula for Trapezoidal Rule

The true error, E_t, in the n-segment trapezoidal rule is estimated as

$$E_t \approx \alpha \frac{1}{n^2}$$

$$E_t \approx \frac{C}{n^2} \tag{4}$$

where C is an approximate constant of proportionality.
Since

$$E_t = TV - I_n \tag{5}$$

where

TV = true value

I_n = approximate value using n-segments

Then from Equations (4) and (5),

$$\frac{C}{n^2} \approx TV - I_n \tag{6}$$

If the number of segments is doubled from n to $2n$ in the trapezoidal rule,

$$\frac{C}{(2n)^2} \approx TV - I_{2n} \tag{7}$$

Equations (6) and (7) can be solved simultaneously to get

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3} \tag{8}$$

2. The estimate of $\int_3^{19} f(x)dx$ is given as 1860.9 using the 1-segment trapezoidal rule. Given $f(7) = 20.27$, $f(11) = 45.125$, and $f(14) = 82.23$, the value of the integral using the 2-segment trapezoidal rule would most nearly be

(A) 787.32

(B) 1072.0

(C) 1144.9

(D) 1291.5

Solution

The correct answer is (D).

The 1-segment trapezoidal rule is

$$I \approx (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

$$1860.9 \approx (19-3)\left[\frac{f(3)+f(19)}{2}\right]$$

$$1860.9 \approx (16)\left[\frac{f(3)+f(19)}{2}\right]$$

The 2-segment trapezoidal rule is

$$I \approx \frac{b-a}{2n}\left[f(a)+2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\}+f(b)\right]$$

$$h = \frac{b-a}{n}$$

$$= \frac{19-3}{2}$$

$$= 8$$

$$I \approx \frac{19-3}{2\times 2}\left[f(3)+2f(11)+f(19)\right]$$

$$= \frac{16}{4}\left[f(3)+f(19)\right]+\frac{16}{4}\left(2f(11)\right)$$

$$= \frac{1860.9}{2}+\frac{16}{4}\left(2f(11)\right)$$

$$= \frac{1860.9}{2}+\frac{16}{4}\left(2\times 45.125\right)$$

$$= 1291.5$$

3. The value of $\int_a^b f(x)dx$ using the 1-, 2-, and 4-segment trapezoidal rule is given as 5.3460, 2.7708, and 1.7536, respectively. The best estimate of the integral you can find using Romberg integration is most nearly

(A) 1.3355
(B) 1.3813
(C) 1.4145
(D) 1.9124

Solution
The correct answer is (B).

$$I_{1,1} = 5.3460$$
$$I_{1,2} = 2.7708$$
$$I_{1,3} = 1.7536$$

where the above three values correspond to using the 1-, 2- and 4-segment trapezoidal rule, respectively. To get the first order extrapolation values,

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, \ k \geq 2$$

$$\begin{aligned} I_{2,1} &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3} \\ &= 2.7708 + \frac{2.7708 - 5.3460}{3} \\ &= 1.9124 \end{aligned}$$

Similarly

$$\begin{aligned} I_{2,2} &= I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3} \\ &= 1.7536 + \frac{1.7536 - 2.7708}{3} \\ &= 1.4145 \end{aligned}$$

For the second order extrapolation value,

$$\begin{aligned} I_{3,1} &= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} \\ &= 1.4145 + \frac{1.4145 - 1.9124}{15} \\ &= 1.3813 \end{aligned}$$

4. Without using the formula for the 1-segment trapezoidal rule for estimating $\int_a^b f(x)dx$, the true error E_t can be found directly as well as exactly by using the formula

$$E_t = -\frac{(b-a)^3}{12} f''(\xi),\ a \le \xi \le b$$

for

(A) $f(x) = e^x$

(B) $f(x) = x^3 + 3x$

(C) $f(x) = 5x^2 + 3$

(D) $f(x) = 5x^2 + e^x$

Solution

The correct answer is (C).

For

$$f(x) = 5x^2 + 3$$
$$f'(x) = 10x$$
$$f''(x) = 10$$

Hence $f''(\zeta) = 10$ irrespective of the value of ζ. Hence, for this function, the true error can be found exactly, that is,

$$E_t = -\frac{(b-a)^3}{12} \times 10$$
$$= -\frac{5}{6}(b-a)^3$$

5. For $\int_a^b f(x)dx$, the true error E_t in the 1-segment trapezoidal rule is given by

$$E_t = -\frac{(b-a)^3}{12} f''(\xi), \quad a \le \xi \le b$$

The value of ξ for $\int_{2.5}^{7.2} 3e^{0.2x} dx$ is most nearly

(A) 2.7998
(B) 4.8500
(C) 4.9601
(D) 5.0327

Solution
The correct answer is (C).

The estimation for the 1-segment trapezoidal rule is

$$\begin{aligned} I &\approx (b-a)\left[\frac{f(a)+f(b)}{2}\right] \\ &= (7.2-2.5)\left[\frac{3e^{0.2\times 2.5} + 3e^{0.2\times 7.2}}{2}\right] \\ &= 41.379 \end{aligned}$$

The true value of the integral is

$$\begin{aligned} \int_{2.5}^{7.2} 3e^{0.2x} dx &= \left[\frac{3e^{0.2x}}{0.2}\right]_{2.5}^{7.2} \\ &= \frac{3e^{0.2\times 7.2}}{0.2} - \frac{3e^{0.2\times 2.5}}{0.2} \\ &= 38.580 \end{aligned}$$

Thus,

$$\begin{aligned} E_t &= 38.580 - 41.379 \\ &= -2.7998 \end{aligned}$$

$$f(x) = 3e^{0.2x}$$

$$f'(x) = 0.6e^{0.2x}$$

$$f''(x) = 0.12e^{0.2x}$$

$$f''(\zeta) = 0.12e^{0.2\zeta}$$

From

$$E_t = -\frac{(b-a)^3}{12} f''(\zeta)$$

we get

$$-2.7998 = -\frac{(7.2-2.5)^3}{12} f''(\xi)$$
$$-2.7998 = -8.65192 \times 0.12 e^{0.2\xi}$$
$$0.32360 = 0.12 e^{0.2\xi}$$
$$2.6968 = e^{0.2\xi}$$
$$\ln(2.69671) = \ln\left(e^{0.2\xi}\right)$$
$$0.99202 = 0.2\xi$$
$$\xi = 4.9601$$

6. The following data of the velocity of a body is given as a function of time.

t (s)	2	4	6	8	10	25
v (m/s)	0.166	0.55115	1.8299	6.0755	20.172	8137.5

The best estimate for the distance in meters covered between $t = 2$ s and $t = 10$ s by using the Romberg rule based on trapezoidal rule results would be

(A) 33.456

(B) 36.877

(C) 37.251

(D) 81.350

Solution

The correct answer is (A).

The estimate for the 1-segment trapezoidal rule is

$$I_{1,1} \approx (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

$$= (10-2)\left[\frac{0.166+20.172}{2}\right]$$

$$= 81.352$$

The estimate for the 2-segment trapezoidal rule is

$$I_{1,2} = \frac{10-2}{2\times 2}\left[0.166+2(1.8299)+20.172\right]$$

$$= 47.996$$

The estimate for the 4-segment trapezoidal rule is

$$I_{1,3} = \frac{10-2}{2\times 4}\left[0.166+2(0.55115+1.8299+6.0755)+20.172\right]$$

$$= 37.251$$

To get the first order extrapolation values,

$$I_{2,1} = I_{1,2} + \frac{I_{1,2}-I_{1,1}}{3}$$

$$= 47.996 + \frac{47.996-81.352}{3}$$

$$= 36.877\,\text{m}$$

Similarly

$$I_{2,2} = I_{1,3} + \frac{I_{1,3}-I_{1,2}}{3}$$

$$= 37.251 + \frac{37.251 - 47.996}{3}$$

$$= 33.670 \text{ m}$$

For the second order extrapolation values,

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}$$

$$= 33.670 + \frac{33.670 - 36.877}{15}$$

$$= 33.456 \text{ m}$$