Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Finite Difference Method
Ordinary Differential Equations
COMPLETE SOLUTION SET

1. The exact solution to the boundary value problem
$$\frac{d^2y}{dx^2} = 6x - 0.5x^2,\ y(0) = 0,\ y(12) = 0$$
for $y(4)$ is
   (A) -234.67
   (B) 0.00
   (C) 16.000
   (D) 37.333

Solution
The correct answer is (A).

First separate the variables.
$$\frac{d^2y}{dx^2} = 6x - 0.5x^2$$
$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 6x - 0.5x^2$$
$$\frac{dy}{dx} = \int \left(6x - 0.5x^2\right)dx$$
$$= \frac{6x^2}{2} - \frac{0.5x^3}{3} + C$$
$$= 3x^2 - \frac{0.5}{3}x^3 + C$$
$$y = \int \left(3x^2 - \frac{0.5}{3}x^3 + C\right)dx$$
$$= \frac{3}{3}x^3 - \frac{0.5}{12}x^4 + Cx + D$$
$$= x^3 - \frac{0.5}{12}x^4 + Cx + D$$

Set $y(0) = 0$ to solve for *D*.

$$y(0) = 0^3 - \frac{0.5}{12} \times 0^4 + C \times 0 + D$$

$$0 = D$$

Set $y(12) = 0$ to solve for C.

$$y(12) = 12^3 - \frac{0.5}{12} \times 12^4 + C \times 12$$

$$0 = 1728 - 864 + 12C$$

$$-12C = 864$$

$$C = -72$$

Thus,

$$y = x^3 - \frac{0.5}{12} x^4 + Cx + D$$

$$= x^3 - \frac{0.5}{12} x^4 - 72x$$

$$y(4) = 4^3 - \frac{0.5}{12} \times 4^4 - 72 \times 4$$

$$= 64 - 10.6667 - 288$$

$$= -234.67$$

2. Given

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2,\ y(0) = 0,\ y(12) = 0$$

the value of $\frac{d^2y}{dx^2}$ at $y(4)$ using the finite difference method and a step size of $h = 4$ can be approximated by

(A) $\frac{y(8) - y(0)}{8}$

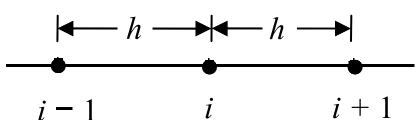
(B) $\frac{y(8) - 2y(4) + y(0)}{16}$

(C) $\frac{y(12) - 2y(8) + y(4)}{16}$

(D) $\frac{y(4) - y(0)}{4}$

Solution

The correct answer is (B).



The finite difference approximation for $\frac{d^2y}{dx^2}$ at node i, that is $x = x_i$, is given by

$$\left.\frac{d^2y}{dx^2}\right|_i \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

where

$$y_{i+1} = y(8)$$
$$y_i = y(4)$$
$$y_{i-1} = y(0)$$
$$h = 4$$

Thus

$$\frac{d^2y}{dx^2} \approx \frac{y(8) - 2y(4) + y(0)}{(4)^2}$$
$$\approx \frac{y(8) - 2y(4) + y(0)}{16}$$

3. Given

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2,\ y(0) = 0,\ y(12) = 0$$

the value of $y(4)$ using the finite difference method with a second order accurate central divided difference method and a step size of $h = 4$ is

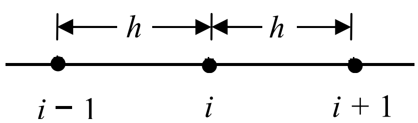
(A) 0.000
(B) 37.333
(C) -234.67
(D) -256.00

Solution

The correct answer is (D).

$$\frac{d^2y}{dx^2} - 6x - 0.5x^2 = 0$$

now



The finite difference approximation for $\frac{d^2y}{dx^2}$ at node i, that is $x = x_i$, is given by

$$\left.\frac{d^2y}{dx^2}\right|_i \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

Thus

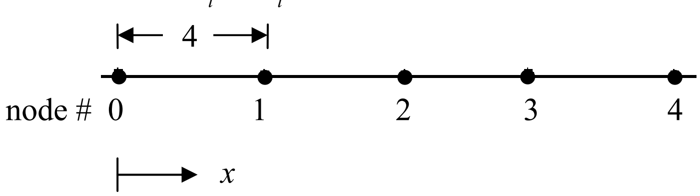
$$\frac{y_{i+1} - 2y_i + y_{i-1}}{(h)^2} - 6x_i + 0.5x_i^2 = 0$$

$$y_{i+1} - 2y_i + y_{i-1} - 6x_ih^2 + 0.5x_i^2h^2 = 0$$

$$y_{i+1} - 2y_i + y_{i-1} = 6x_ih^2 - 0.5x_i^2h^2$$

$$= 6x_i4^2 - 0.5x_i^24^2$$

$$= 96x_i - 8x_i^2$$



At node $i=0, x_0=0$

$$y_0 = 0$$

At node $i=1, x_1=4$

$$\begin{aligned} y_2 - 2y_1 + y_0 &= 96x_1 - 8x_1^2 \\ &= 96(4) - 8(16) \\ &= 256 \end{aligned}$$

At node $i=2, x_2=8$

$$\begin{aligned} y_3 - 2y_2 + y_1 &= 96x_2 - 8x_2^2 \\ &= 96(8) - 8(64) \\ &= 256 \end{aligned}$$

At node $i=3, x_3=12$

$$y_3 = 0$$

Writing the above equations in matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 256 \\ 256 \\ 0 \end{bmatrix}$$

which gives

$$\begin{aligned} y(0) &= y_0 = 0 \\ y(4) &\approx y_1 = -256 \\ y(8) &\approx y_2 = -256 \\ y(12) &= y_3 = 0 \end{aligned}$$

Hence

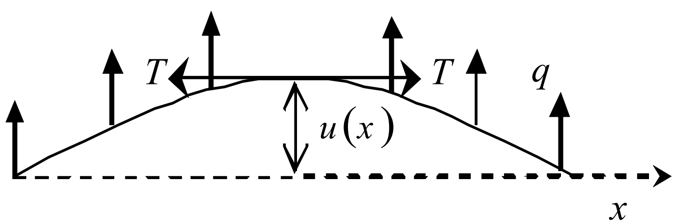
$$y(4) \approx -256$$

4. The transverse deflection u of a cable of length L that is fixed at both ends, is given as a solution to

$$\frac{d^2u}{dx^2} = \frac{Tu}{R} + \frac{qx(x-L)}{2R}$$

where

T = tension in cable
R = flexural stiffness
q = distributed transverse load



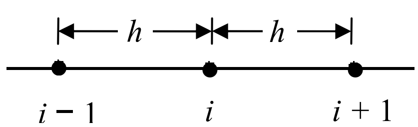
Given $L = 50''$, $T = 2000\ \text{lbs}$, $q = 75\ \frac{\text{lbs}}{\text{in}}$, and $R = 75\times10^6\ \text{lbs}\cdot\text{in}^2$

Using finite difference method modeling with second order central divided difference accuracy and a step size of $h = 12.5''$, the value of the deflection at the center of the cable most nearly is

(A) 0.072737″
(B) 0.080832″
(C) 0.081380″
(D) 0.084843″

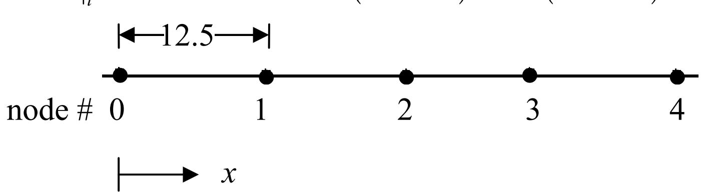
Solution

The correct answer is (D).



At node i, using finite difference approximations,

$$\left.\frac{d^2u}{dx^2}\right|_i \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = \frac{(2000)u_i}{(75\times10^6)} + \frac{75\times x_i(x_i - 50)}{2(75\times10^6)}$$



For $i=0,\ x_0=0$

$$u_0 = 0'' \tag{1}$$

For $i=1,\ x_1=12.5$

$$\frac{u_2-2u_1+u_0}{h^2}=\frac{(2000)u_1}{(75\times10^6)}+\frac{75\times x_1(x_1-50)}{2(75\times10^6)}$$

$$\frac{u_2-2u_1+u_0}{12.5^2}=\frac{(2000)u_1}{(75\times10^6)}+\frac{75\times12.5(12.5-50)}{2(75\times10^6)}$$

$$\frac{u_2-2u_1+u_0}{156.25}=2.6667\times10^{-5}u_1-2.3438\times10^{-4}$$

$$u_2-2u_1+u_0=4.1667\times10^{-3}u_1-3.6621\times10^{-2}$$

$$u_2-2.0042u_1+u_0=-3.6621\times10^{-2} \tag{2}$$

For $i=2,\ x_2=25$

$$\frac{u_3-2u_2+u_1}{h^2}=\frac{(2000)u_2}{(75\times10^6)}+\frac{75\times x_2(x_2-50)}{2(75\times10^6)}$$

$$\frac{u_3-2u_2+u_1}{12.5^2}=\frac{(2000)u_2}{(75\times10^6)}+\frac{75\times25(25-50)}{2(75\times10^6)}$$

$$\frac{u_3-2u_2+u_1}{156.25}=2.6667\times10^{-5}u_2-3.1250\times10^{-4}$$

$$u_3-2u_2+u_1=4.1667\times10^{-3}u_2-4.8828\times10^{-2}$$

$$u_3-2.0042u_2+u_1=-4.8828\times10^{-2} \tag{3}$$

For $i=3,\ x_3=37.5$

$$\frac{u_4-2u_3+u_2}{h^2}=\frac{(2000)u_3}{(75\times10^6)}+\frac{75\times x_3(x_3-50)}{2(75\times10^6)}$$

$$\frac{u_4-2u_3+u_2}{12.5^2}=\frac{(2000)u_3}{(75\times10^6)}+\frac{75\times37.5(37.5-50)}{2(75\times10^6)}$$

$$\frac{u_4-2u_3+u_2}{156.25}=2.6667\times10^{-5}u_3-2.3438\times10^{-4}$$

$$u_4-2u_3+u_2=4.1667\times10^{-3}u_3-3.6621\times10^{-2}$$

$$u_4-2.0042u_3+u_2=-3.6621\times10^{-2} \tag{4}$$

For $i=4,\ x_4=50$

$$u_4 = 0'' \tag{5}$$

Rewrite Equations 1-5 in matrix form

$$\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 1 & -2.0042 & 1 & 0 & 0\\ 0 & 1 & -2.0042 & 1 & 0\\ 0 & 0 & 1 & -2.0042 & 1\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}u_0\\ u_1\\ u_2\\ u_3\\ u_4\end{bmatrix}=\begin{bmatrix}0\\ -3.6621\times10^{-2}\\ -4.8828\times10^{-2}\\ -3.6621\times10^{-2}\\ 0\end{bmatrix}$$

Solving this matrix by using Gauss elimination gives

$$u_0 = 0''$$

$$u_1 = 0.060605''$$

$$u_2 = 0.084843''$$

$$u_3 = 0.060605''$$

$$u_4 = 0''$$

Hence

$$u(25) \approx u_2 = 0.084843''$$

5. The radial displacement u of a pressurized hollow thick cylinder (inner radius = 5″, outer radius = 8″) is given at different radial locations.

Radius (in)	Radial Displacement (in)
5.0	0.0038731
5.6	0.0036165
6.2	0.0034222
6.8	0.0032743
7.4	0.0031618
8.0	0.0030769

The maximum normal stress, in psi, on the cylinder is given by

$$\sigma_{\max} = 3.2967 \times 10^6 \left(\frac{u(5)}{5} + 0.3 \frac{du}{dr}(5) \right)$$

The maximum stress, in psi, with second order accuracy is

(A) 2079.6
(B) 2104.5
(C) 2130.7
(D) 2182.0

Solution

The correct answer is (A).

If we look at the Taylor series

$$u(r+\Delta r) = u(r) + u'(r)\Delta r + \frac{u''(r)(\Delta r)^2}{2} + \frac{u'''(r)(\Delta r)^3}{6} + \cdots$$

and

$$u(r+2\Delta r) = u(r) + u'(r)(2\Delta r) + \frac{u''(r)(2\Delta r^2)}{2} + \frac{u'''(r)(2\Delta r)^3}{6} + \cdots$$

Multiply the first equation by 4 and subtract it from the second equation.

$$u(r+2\Delta r) - 4u(r+\Delta r) = -3u(r) - u'(r)2\Delta r + O(\Delta r)^3$$

$$u'(r)(2\Delta r) = -u(r+2\Delta r) + 4u(r+\Delta r) - 3u(r) + O(\Delta r)^3$$

$$u'(r) = \frac{-u(r+2\Delta r) + 4u(r+\Delta r) - 3u(r)}{2\Delta r} + \frac{O(\Delta r)^3}{2\Delta r}$$

$$u'(r) = \frac{-u(r+2\Delta r) + 4u(r+\Delta r) - 3u(r)}{2\Delta r} + O(\Delta r)^2$$

This equation is second order accurate, that is, the true error is $O(\Delta r)^2$.

$$\frac{du}{dr} \approx \frac{-u(r+2\Delta r) + 4u(r+\Delta r) - 3u(r)}{2\Delta r}$$

$$r = 5,\ \Delta r = 0.6$$

$$\frac{du}{dr}(5) \approx \frac{-u(5 + 2\times 0.6) + 4u(5 + 0.6) - 3u(5)}{2\times 0.6}$$

$$= \frac{-u(6.2) + 4u(5.6) - 3u(5)}{2\times 0.6}$$

$$= \frac{-0.0034222 + 0.014466 - 0.011619}{1.2}$$

$$= -0.00047933$$

Thus,

$$\sigma_{\max} = 3.2967\times 10^{6}\left(\frac{u(5)}{5} + 0.3\frac{du}{dr}(5)\right)$$

$$\approx 3.2967\times 10^{6}\left(\frac{0.0038731}{5} + 0.3\times(-0.00047933)\right)$$

$$= 2079.6 \text{ psi}$$

6. For a simply supported beam (at $x = 0$ and $x = L$) with a uniform load q, the vertical deflection $v(x)$ is described by the boundary value ordinary differential equation as

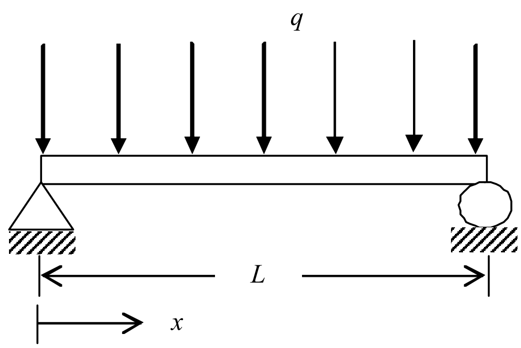
$$\frac{d^2v}{dx^2} = \frac{qx(x-L)}{2EI}, \quad 0 \le x \le L$$

where

E = Young's modulus of the beam

I = second moment of area

This ordinary differential equation is based on assuming that $\frac{dv}{dx}$ is small. If $\frac{dv}{dx}$ is not small, then the ordinary differential equation is given by



(A) $$\frac{\frac{d^2v}{dx^2}}{\sqrt{1+\left(\frac{dv}{dx}\right)^2}} = \frac{qx(x-L)}{2EI}$$

(B) $$\frac{\frac{d^2v}{dx^2}}{\left(1+\left(\frac{dv}{dx}\right)^2\right)^{3/2}} = \frac{qx(x-L)}{2EI}$$

(C) $$\frac{\frac{d^2v}{dx^2}}{\sqrt{1+\left(\frac{dv}{dx}\right)}} = \frac{qx(x-L)}{2EI}$$

(D) $$\frac{\frac{d^2v}{dx^2}}{1+\frac{dv}{dx}} = \frac{qx(x-L)}{2EI}$$

Solution

The correct answer is (B).

The equation for the deflection in a beam is

$$\frac{1}{\rho} = \frac{M}{EI}$$

Where

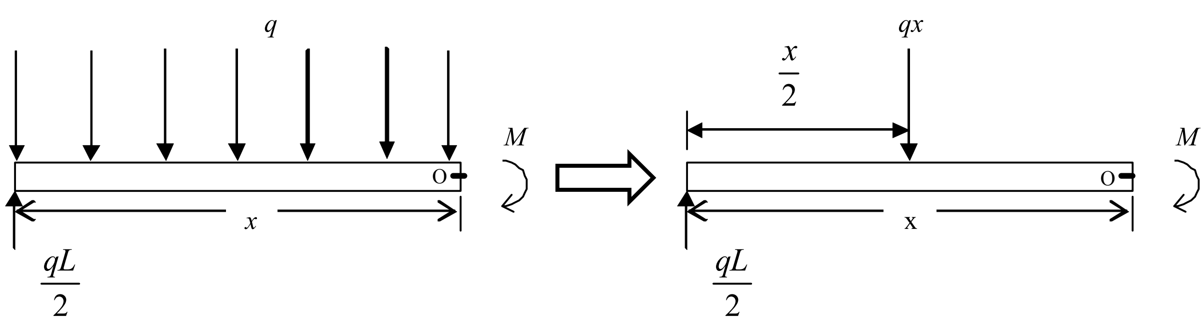
$$\frac{1}{\rho} = \text{curvature of the beam}$$

$M =$ internal moment of the beam where the curvature is to be determined

The curvature of a function v can be rewritten in rectangular format as

$$\frac{1}{\rho} = \frac{\dfrac{d^2v}{dx^2}}{\left(1+\left(\dfrac{dv}{dx}\right)^2\right)^{3/2}}$$

The reaction at each support is $\frac{qL}{2}$.



Writing the balance of the bending moments at point O at a distance *x* from the left end,

$$M + \frac{qL}{2}x - (qx)\frac{x}{2} = 0$$

$$M = \frac{qx^2}{2} - \frac{qL}{2}x$$

$$= \frac{qx(x-L)}{2}$$

Thus the equation for the deflection of the simply supported beam is

$$\frac{\dfrac{d^2v}{dx^2}}{\left(1+\left(\dfrac{dv}{dx}\right)^2\right)^{3/2}} = \frac{qx(x-L)}{2EI}$$