Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Runge-Kutta 4th Order Method
Ordinary Differential Equations
COMPLETE SOLUTION SET

1. To solve the ordinary differential equation

$$3\frac{dy}{dx} + xy^2 = \sin x,\ y(0) = 5,$$

by Runge-Kutta 4th order method, you need to rewrite the equation as

(A) $\frac{dy}{dx} = \sin x - xy^2,\ y(0) = 5$

(B) $\frac{dy}{dx} = \frac{1}{3}\left(\sin x - xy^2\right),\ y(0) = 5$

(C) $\frac{dy}{dx} = \frac{1}{3}\left(-\cos x - \frac{xy^3}{3}\right),\ y(0) = 5$

(D) $\frac{dy}{dx} = \frac{1}{3}\sin x,\ y(0) = 5$

Solution

The correct answer is (B)

$$3\frac{dy}{dx} + xy^2 = \sin x,\ y(0) = 5$$

is rewritten as

$$3\frac{dy}{dx} = \sin x - xy^2,\ y(0) = 5$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\sin x - xy^2\right),\ y(0) = 5$$

2. Given $3\frac{dy}{dx}+5y^2=\sin x,\ y(0.3)=5$ and using a step size of $h=0.3$, the value of $y(0.9)$ using Runge-Kutta 4th order method is most nearly

(A) -0.25011×10^{40}
(B) -4297.4
(C) -1261.5
(D) 0.88498

Solution

The correct answer is (C)

$$3\frac{dy}{dx}+5y^2=\sin x,\ y(0.3)=5$$

is rewritten as

$$\frac{dy}{dx}=\frac{1}{3}\left(\sin x-5y^2\right),\ y(0.3)=5$$

the Runge-Kutta 4th order method is

$$y_{i+1}=y_i+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$$

where

$$k_1=f(x_i,y_i)$$
$$k_2=f\left(x_i+\frac{1}{2}h,y_i+\frac{1}{2}k_1h\right)$$
$$k_3=f\left(x_i+\frac{1}{2}h,y_i+\frac{1}{2}k_2h\right)$$
$$k_4=f(x_i+h,y_i+k_3h)$$

for $i=0,\ x_0=0.3, y_0=5$

$$k_1=f(x_0,y_0)$$
$$=f(0.3,5)$$
$$=\frac{1}{3}\left(\sin 0.3-5(5)^2\right)$$
$$=-41.5682$$
$$k_2=f\left(x_0+\frac{1}{2}h,y_0+\frac{1}{2}k_1h\right)$$
$$=f\left(0.3+\frac{1}{2}\times0.3,5+\frac{1}{2}\times-41.5682\times0.3\right)$$
$$=f(0.45,-1.23523)$$
$$=\frac{1}{3}\left(\sin 0.45-5(-1.23523)^2\right)$$

$$= -2.398$$

$$k_3 = f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2 h\right)$$

$$= f\left(0.3 + \frac{1}{2} \times 0.3, 5 + \frac{1}{2} \times -2.398 \times 0.3\right)$$

$$= f(0.45, 4.6403)$$

$$= \frac{1}{3}\left(\sin 0.45 - 5(4.6403)^2\right)$$

$$= -35.7423$$

$$k_4 = f(x_0 + h, y_0 + k_3 h)$$

$$= f(0.3 + 0.3, 5 + -35.7423 \times 0.3)$$

$$= f(0.6, -5.72269)$$

$$= \frac{1}{3}(\sin 0.6 - 5(-5.72269))$$

$$= -54.3938$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 5 + \frac{1}{6}(-41.5682 - 2 \times 2.398 - 2 \times 35.7423 - 54.3938) \times 0.3$$

$$= -3.61213$$

$$x_1 = x_0 + h$$

$$= 0.3 + 0.3$$

$$= 0.6$$

for $i = 1,\ x_1 = 0.6, y_1 = -3.61213$

$$k_1 = f(x_1, y_1)$$

$$= f(0.6, -3.61213)$$

$$= \frac{1}{3}\left(\sin 0.6 - 5(-3.61213)^2\right)$$

$$= -21.5576$$

$$k_2 = f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1 h\right)$$

$$= f\left(0.6 + \frac{1}{2} \times 0.3, -3.61213 + \frac{1}{2} \times -21.5576 \times 0.3\right)$$

$$= f(0.75, -6.84577)$$

$$= \frac{1}{3}\left(\sin 0.75 - 5(-6.84577)^2\right)$$

$$= -77.8804$$

$$k_3 = f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2 h\right)$$
$$= f\left(0.6 + \frac{1}{2} \times 0.3, -3.61213 + \frac{1}{2} \times -77.8804 \times 0.3\right)$$
$$= f(0.75, -15.2942)$$
$$= \frac{1}{3}\left(\sin 0.75 - 5(-15.2942)^2\right)$$
$$= -389.627$$
$$k_4 = f(x_0 + h, y_0 + k_3 h)$$
$$= f(0.6 + 0.3, -3.61213 + -389.627 \times 0.3)$$
$$= f(0.9, -120.5)$$
$$= \frac{1}{3}(\sin 0.9 - 5(-120.5))$$
$$= -24200.2$$
$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$
$$= -3.61213 + \frac{1}{6}(-21.5576 - 2 \times 77.8804 - 2 \times 389.627 - 24200.2) \times 0.3$$
$$= -1261.45$$
$$x_2 = x_1 + h$$
$$= 0.6 + 0.3$$
$$= 0.9$$
$$y(x_2) \cong -1261.45$$
$$y(0.9) \cong -1261.45$$

3. Given $3\frac{dy}{dx}+y^2=e^x,\ y(0.3)=5$, and using a step size of $h=0.3$, the best estimate of $\frac{dy}{dx}(0.9)$ Runge-Kutta 4th order method is most nearly

(A) -1.6604
(B) -1.1785
(C) -0.45831
(D) 2.7270

Solution

The correct answer is (A)

$$3\frac{dy}{dx}+y^2=e^x,\ y(0.3)=5$$

is rewritten as

$$\frac{dy}{dx}=\frac{1}{3}\left(e^x-y^2\right),\ y(0.3)=5$$

the Runge-Kutta 4th order method is

$$y_{i+1}=y_i+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$$

where

$$k_1=f(x_i,y_i)$$

$$k_2=f\left(x_i+\frac{1}{2}h,y_i+\frac{1}{2}k_1h\right)$$

$$k_3=f\left(x_i+\frac{1}{2}h,y_i+\frac{1}{2}k_2h\right)$$

$$k_4=f(x_i+h,y_i+k_3h)$$

for $i=0,\ x_0=0.3, y_0=5$

$$\begin{aligned}k_1&=f(x_0,y_0)\\&=f(0.3,5)\\&=\frac{1}{3}\left(e^{0.3}-(5)^2\right)\\&=-7.8833\end{aligned}$$

$$\begin{aligned}k_2&=f\left(x_0+\frac{1}{2}h,y_0+\frac{1}{2}k_1h\right)\\&=f\left(0.3+\frac{1}{2}\times0.3,5+\frac{1}{2}\times-7.8833\times0.3\right)\\&=f(0.45,3.81751)\\&=\frac{1}{3}\left(e^{0.45}-(3.81751)^2\right)\\&=-4.33502\end{aligned}$$

$$
\begin{aligned}
k_3 &= f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2 h\right) \\
&= f\left(0.3 + \frac{1}{2}\times 0.3, 5 + \frac{1}{2}\times -4.33502 \times 0.3\right) \\
&= f(0.45, 4.34975) \\
&= \frac{1}{3}\left(e^{0.45} - 4.34975^2\right) \\
&= -5.784 \\
k_4 &= f(x_0 + h, y_0 + k_3 h) \\
&= f(0.3 + 0.3, 5 + -5.784 \times 0.3) \\
&= f(0.6, 3.2648) \\
&= \frac{1}{3}\left(e^{0.6} - 3.2648^2\right) \\
&= -2.9456 \\
y_1 &= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\
&= 5 + \frac{1}{6}(-7.88338 - 2\times 4.33502 - 2\times 5.784 - 2.9456)\times 0.3 \\
&= 3.44665
\end{aligned}
$$

for $i = 1,\ x_1 = 0.6, y_1 = 3.44665$

$$
\begin{aligned}
k_1 &= f(x_1, y_1) \\
&= f(0.6, 3.44665) \\
&= \frac{1}{3}\left(e^{0.6} - 3.44665^2\right) \\
&= -3.35243 \\
k_2 &= f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1 h\right) \\
&= f\left(0.6 + \frac{1}{2}\times 0.3, 3.44665 + \frac{1}{2}\times -3.35243 \times 0.3\right) \\
&= f(0.75, 2.94378) \\
&= \frac{1}{3}\left(e^{0.75} - (2.94378)^2\right) \\
&= -2.18295
\end{aligned}
$$

$$k_3 = f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2 h\right)$$

$$= f\left(0.6 + \frac{1}{2}\times 0.3, 3.44665 + \frac{1}{2}\times -2.18295 \times 0.3\right)$$

$$= f(0.75, 3.11921)$$

$$= \frac{1}{3}\left(e^{0.75} - (3.11921)^2\right)$$

$$= -2.53749$$

$$k_4 = f(x_0 + h, y_0 + k_3 h)$$

$$= f(0.6 + 0.3, 3.44665 - 2.53749 \times 0.3)$$

$$= f(0.9, 2.6854)$$

$$= \frac{1}{3}\left(e^{0.9} - (2.6854)^2\right)$$

$$= -1.58392$$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 3.44665 + \frac{1}{6}(-3.35243 - 2\times 2.18295 - 2\times 2.53749 - 1.58392)\times 0.3$$

$$= 2.72779$$

$$x_2 = x_1 + h$$

$$= 0.6 + 0.3$$

$$= 0.9$$

$$\frac{dy}{dx} = \frac{1}{3}\left(e^x - y^2\right)$$

$$\frac{dy}{dx}(0.9) \cong \frac{1}{3}\left(e^{0.9} - (y(0.9))^2\right)$$

$$\cong \frac{1}{3}\left(e^{0.9} - (2.72779)^2\right)$$

$$\cong -1.6604$$

4. The velocity (m/s) of a parachutist is given as a function of time (seconds) by

$$v(t) = 55.8\tanh(0.17t), t \ge 0$$

Using Runge-Kutta 4[th] order method with a step size of 5 seconds, the distance traveled by the body from $t = 2$ to $t = 12$ seconds is estimated most nearly as

(A) 341.43 m
(B) 428.97 m
(C) 429.05 m
(D) 703.50 m

Solution

The correct answer is (C)

$$v(t) = 55.8\tanh(0.17t), t \ge 0$$

the Runge-Kutta 4[th] order method is

$$S_{i+1} = S_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

where

$$k_1 = f(t_i, S_i)$$

$$k_2 = f\left(t_i + \frac{1}{2}h, S_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(t_i + \frac{1}{2}h, S_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f(t_i + h, S_i + k_3 h)$$

for $i = 0,\ t_0 = 2, S_0 = 0$

$$\begin{aligned} k_1 &= f(t_0, S_0) \\ &= f(2,0) \\ &= 55.8\tanh(0.17 \times 2) \\ &= 18.2732 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(t_0 + \frac{1}{2}h, S_0 + \frac{1}{2}k_1 h\right) \\ &= f\left(2 + \frac{1}{2} \times 5, 0 + \frac{1}{2} \times 18.2732 \times 5\right) \\ &= f(4.5, 45.683) \\ &= 55.8\tanh(0.17 \times 4.5) \\ &= 35.9359 \end{aligned}$$

$$\begin{aligned} k_3 &= f\left(t_0 + \frac{1}{2}h, S_0 + \frac{1}{2}k_2 h\right) \\ &= f\left(2 + \frac{1}{2} \times 5, 0 + \frac{1}{2} \times 35.9359 \times 5\right) \end{aligned}$$

$$
\begin{aligned}
&= f(4.5, 89.8398) \\
&= 55.8\tanh(0.17 \times 4.5) \\
&= 35.9359
\end{aligned}
$$

$$
\begin{aligned}
k_4 &= f(t_0 + h, S_0 + k_3 h) \\
&= f(2 + 5, 0 + 35.9359 \times 5) \\
&= f(7, 179.68) \\
&= 55.8\tanh(0.17 \times 7) \\
&= 46.3463
\end{aligned}
$$

$$
\begin{aligned}
S_1 &= S_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\
&= 0 + \frac{1}{6}(18.2732 + 2 \times 35.9359 + 2 \times 35.9359 + 46.3463) \times 5 \\
&= 173.636
\end{aligned}
$$

$$
\begin{aligned}
t_1 &= t_0 + h \\
&= 2 + 5 \\
&= 7
\end{aligned}
$$

for $i = 1,\ t_1 = 7, S_1 = 173.636$

$$
\begin{aligned}
k_1 &= f(t_1, S_1) \\
&= f(7, 173.636) \\
&= 55.8\tanh(0.17 \times 7) \\
&= 46.3463
\end{aligned}
$$

$$
\begin{aligned}
k_2 &= f\left(t_1 + \frac{1}{2}h, S_1 + \frac{1}{2}k_1 h\right) \\
&= f\left(7 + \frac{1}{2} \times 5, 173.636 + \frac{1}{2} \times 46.3463 \times 5\right) \\
&= f(9.5, 289.502) \\
&= 55.8\tanh(0.17 \times 9.5) \\
&= 51.5534
\end{aligned}
$$

$$
\begin{aligned}
k_3 &= f\left(t_1 + \frac{1}{2}h, S_1 + \frac{1}{2}k_2 h\right) \\
&= f\left(7 + \frac{1}{2} \times 5, 173.636 + \frac{1}{2} \times 51.5534 \times 5\right) \\
&= f(9.5, 302.52) \\
&= 55.8\tanh(0.17 \times 9.5) \\
&= 51.5534
\end{aligned}
$$

$$k_4 = f(t_1 + h, S_1 + k_3 h)$$
$$= f(7 + 5, 173.636 + 51.5534 \times 5)$$
$$= f(12, 431.403)$$
$$= 55.8 \tanh(0.17 \times 12)$$
$$= 53.9445$$

$$S_2 = S_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$
$$= 173.636 + \frac{1}{6}(46.3463 + 2 \times 51.5534 + 2 \times 51.5534 + 53.9445) \times 5$$
$$= 429.05\, m$$

$$t_2 = t_1 + h$$
$$= 7 + 5$$
$$= 12$$

The distance traveled from $t = 2$ to $t = 12$ is

$$\text{Distance traveled} = S_2 - S_0$$
$$= S(t_2) - S(t_0)$$
$$= S(12) - S(2)$$
$$= 429.05 - 0$$
$$= 429.05 m$$

5. Runge-Kutta method can be derived from using first three terms of Taylor series of writing the value of y_{i+1}, that is the value of y at x_{i+1}, in terms of y_i and all the derivatives of y at x_i. If $h = x_{i+1} - x_i$, the explicit expression for y_{i+1} if the first five terms of the Taylor series are chosen for the ordinary differential equation

$$\frac{dy}{dx} + 5y = 3e^{-2x},\ y(0) = 7,$$

would be

(A) $$y_{i+1} = y_i + \left(3e^{-2x_i} - 5y_i\right)h + \frac{5h^2}{2}$$

(B) $$y_{i+1} = y_i + \left(3e^{-2x_i} - 5y_i\right)h + \left(-21e^{-2x_i} + 25y_i\right)\frac{h^2}{2} + \left(-483e^{-2x_i} + 625y_i\right)\frac{h^3}{6} + \left(-300909e^{-2x_i} + 390625y_i\right)\frac{h^4}{24}$$

(C) $$y_{i+1} = y_i + \left(3e^{-2x_i} - 5y_i\right)h + \left(-6e^{-2x_i}\right)\frac{h^2}{2} + \left(12e^{-2x_i}\right)\frac{h^3}{6} + \left(-24e^{-2x_i}\right)\frac{h^4}{24}$$

(D) $$y_{i+1} = y_i + \left(3e^{-2x_i} - 5y_i\right)h + \left(-6e^{-2x_i} + 5\right)\frac{h^2}{2} + \left(12e^{-2x_i}\right)\frac{h^3}{6} + \left(-24e^{-2x_i}\right)\frac{h^4}{24}$$

Solution

The correct answer is (B)

The first five terms of the Taylor series are as follows

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2 + \frac{1}{3!}f''(x_i, y_i)h^3 + \frac{1}{4!}f'''(x_i, y_i)h^4$$

our ordinary differential equation

$$\frac{dy}{dx} + 5y = 3e^{-2x},\ y(0) = 7$$

$$f(x, y) = 3e^{-2x} - 5y$$

Now since y is a function of x,

$$f'(x, y) = \frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y}\frac{dy}{dx}$$

$$= \frac{\partial}{\partial x}\left(3e^{-2x} - 5y\right) + \frac{\partial}{\partial y}\left[\left(3e^{-2x} - 5y\right)\right]\left(3e^{-2x} - 5y\right)$$

$$= -6e^{-2x} + (-5)\left(3e^{-2x} - 5y\right)$$
$$= -21e^{-2x} + 25y$$

$$f''(x,y) = \frac{\partial f'(x,y)}{\partial x} + \frac{\partial f'(x,y)}{\partial y}\frac{d^2y}{dx^2}$$
$$= \frac{\partial}{\partial x}\left(-21e^{-2x} + 25y\right) + \frac{\partial}{\partial y}\left[\left(-21e^{-2x} + 25y\right)\right]\left(-21e^{-2x} + 25y\right)$$
$$= 42e^{-2x} + (25)\left(-21e^{-2x} + 25y\right)$$
$$= -483e^{-2x} + 625y$$

$$f'''(x,y) = \frac{\partial f''(x,y)}{\partial x} + \frac{\partial f''(x,y)}{\partial y}\frac{d^3y}{dx^3}$$
$$= \frac{\partial}{\partial x}\left(-483e^{-2x} + 625y\right) + \frac{\partial}{\partial y}\left[\left(-483e^{-2x} + 625y\right)\right]\left(-483e^{-2x} + 625y\right)$$
$$= 966e^{-2x} + (625)\left(-483e^{-2x} + 625y\right)$$
$$= -3000909e^{-2x} + 390625y$$

The 4th order formula for the above example would be

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2 + \frac{1}{3!}f''(x_i, y_i)h^3 + \frac{1}{4!}f'''(x_i, y_i)h^4$$

$$y_{i+1} = y_i + \left(3e^{-2x_i} - 5y_i\right)h + \left(-21e^{-2x_i} + 25y_i\right)\frac{h^2}{2}$$
$$+ \left(-483e^{-2x_i} + 625y_i\right)\frac{h^3}{6} + \left(-300909e^{-2x_i} + 390625y_i\right)\frac{h^4}{24}$$

6. A hot solid cylinder is immersed in an cool oil bath as part of a quenching process. This process makes the temperature of the cylinder, θ_c, and the bath, θ_b, change with time. If the initial temperature of the bar and the oil bath is given as 600° C and 27°C, respectively, and

Length of cylinder = 30 cm

Radius of cylinder = 3 cm

Density of cylinder = 2700 $\frac{kg}{m^3}$

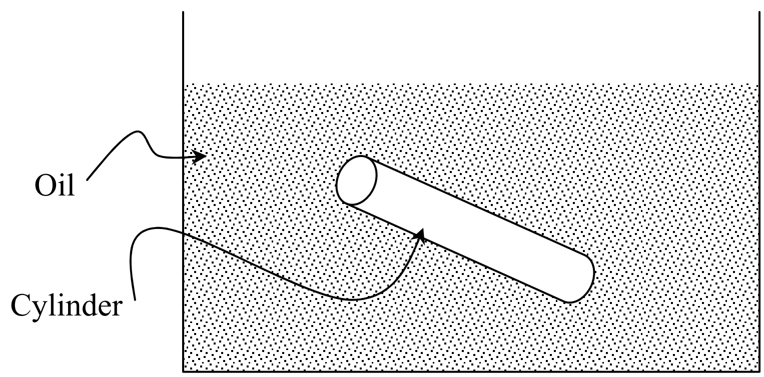
Specific heat of cylinder = 895 $\frac{J}{kg \cdot K}$

Convection heat transfer coefficient = 100 $\frac{W}{m^2 \cdot K}$

Specific heat of oil = 1910 $\frac{J}{kg \cdot K}$

Mass of oil = 2 kg

the coupled ordinary differential equation giving the heat transfer are given by



(A) $$362.4\frac{d\theta_c}{dt} + \theta_c = \theta_b$$

$$675.5\frac{d\theta_b}{dt} + \theta_b = \theta_c$$

(B) $$362.4\frac{d\theta_c}{dt} - \theta_c = \theta_b$$

$$675.5\frac{d\theta_b}{dt} - \theta_b = \theta_c$$

(C) $$675.5\frac{d\theta_c}{dt} + \theta_c = \theta_b$$

$$362.4\frac{d\theta_b}{dt} + \theta_b = \theta_c$$

(B) $$675.5\frac{d\theta_c}{dt} - \theta_c = \theta_b$$

Solution
The correct answer is (A)

For the cylinder the rate of heat lost due to convection $= h(\theta)A(\theta_c - \theta_b)$.

where

h(θ) = the convective cooling coefficient, W/(m^2-K) and is a function of temperature

A = surface area of the cylinder

The energy stored in the mass is given by

Energy stored by mass = m.C.θ

where

m = mass of the cylinder, kg

C = specific heat of the cylinder, J/(kg-K)

From an energy balance,

Rate at which heat is gained - Rate at which heat is lost =

= Rate at which heat is stored

gives

$$-h(\theta)A(\theta_c - \theta_b) = mC\frac{d\theta}{dt}$$

where

$$h(\theta) = 100\ \frac{W}{m^2 \cdot K}$$

$$\begin{aligned}A &= 2\pi r^2 + 2\pi rL \\ &= 2\pi(0.03)^2 + 2\pi \times 0.03 \times 0.3 \\ &= 0.062204\, m^2\end{aligned}$$

$$\begin{aligned}m &= \rho V \\ &= 2700\frac{kg}{m^3} \times (\pi r^2 \times L)m^3 \\ &= 2700 \times \pi(0.03)^2 \times 0.3 \\ &= 2.2902\, kg\end{aligned}$$

Thus

$$-h(\theta)A(\theta_c - \theta_b) = mC\frac{d\theta_c}{dt}$$

$$-100 \times 0.062204(\theta_c - \theta_b) = 2.2902 \times 895\frac{d\theta_c}{dt}$$

$$329.517\frac{d\theta_c}{dt} + \theta_c = \theta_b$$

Similarly for the oil

$$-h(\theta)A(\theta_b - \theta_c) = mC\frac{d\theta_b}{dt}$$

$$-100 \times 0.062204(\theta_b - \theta_c) = 2 \times 1910\frac{d\theta_b}{dt}$$

$$614.108\frac{d\theta_b}{dt} + \theta_b = \theta_c$$