Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Chapter 09.02 Newton's Method
Optimization

COMPLETE SOLUTION SET

1. Which of the following is NOT required for using Newton's method for optimization?
   (A) The lower bound for the search region
   (B) Twice differentiable optimization function
   (C) The function to be optimized
   (D) A good initial estimate that is reasonably close to the optimal

Solution
The correct answer is (A).

Newton's method is not a bracketing method but an open method. Only bracketing methods require a lower (or upper) bound for the search region. Therefore (A) is not required for using Newton's method.

2. Which of the following statements is INCORRECT?

(A) If the second derivative at x_i is negative, then x_i is a maximum.

(B) If the first derivative at x_i is zero, then x_i is an optimum.

(C) If x_i is a maximum, then the second derivative at x_i is positive.

(D) The value of the function can be positive or negative as any optima.

Solution

The correct answer is (C).

When the function is at a maximum, its second derivative has a negative value and not a positive value.

3. For what value of x, is the function $x^2 - 2x - 6$ minimized?
   (A) 0
   (B) 1
   (C) 5
   (D) 3

Solution
The correct answer is (B).

Probably the easiest way to solve the problem is to recognize that $f^{'}(x) = 2x - 2$ is equal to zero when $x = 1$ and the second derivative is positive. So the function is a minimum at $x = 0$, and the correct answer is *(B).* This is also the absolute minimum because the first derivative is zero only at a single point.Using an initial estimate of 0, using Newton's method, the first iteration would converge to the optimal solution as follows.

$$\begin{aligned} x_1 &= x_0 - \frac{f^{'}(x_0)}{f^{"}(x_0)} \\ &= 0 - \frac{2(0) - 2}{2} \\ &= 1 \end{aligned}$$

4. We need to enclose a field with a fence. We have 500 feet of fencing material with a building on one side of the field where we will not need any fencing. Determine the maximum area of the field that can be enclosed by the fence.

(A) $x = 125, y = 250$
(B) $x = 150, y = 200$
(C) $x = 125, y = 100$
(D) $x = 200, y = 150$

Solution
The correct answer is (A).
Let x and y represent the two edges of the rectangular area to be fenced. We can then represent the relationship of material availability as:

$$2x + y = 500$$

The function to maximize is simply the area of the rectangular shape which is

$$A = xy$$

Substituting the value of y in terms of x, we obtain the following one-dimensional optimization function

$$f(x) = x(500 - 2x) = 500x - 2x^2$$

where

$$f'(x) = -4x + 500$$

Once again the solution is fairly obvious at this stage and it is $x = 125$ where $f'(x) = 0$. Using an initial estimate of 0, using Newton's method, the first iteration would converge to the optimal solution as follows.

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)}$$

$$= 0 - \frac{-4(0) + 500}{-4}$$

$$= 125$$

Using equation the $2x + y = 500$ and $x = 125$,we can determine $y = 250$ .Therefore, the correct answer is (A). The maximum fenced-in area is 31250 square feet.

5. A rectangular box with a square base and no top has a volume of 500 cubic inches. Find the length of the edge of the square base and height for the box that requires the least amount of material to build. Conduct two iterations using an initial guess of $l = 5\,in$.

(A) Base edge length is 10.00 and height is 5.00
(B) Base edge length is 9.17 and height is 6.00
(C) Base edge length is 9.00 and height is 6.17
(D) Base edge length is 10.00 and height is 10.00

Solution

The correct answer is (C).

If we let h represent the height of the box and l the length of the edge of the square base. The volume of the container V can be written as

$$V = 500 = l^2 h.$$

The box that requires the least amount of material to build is also the box with the smallest surface area. Therefore, the function to be minimized is

$$A = l^2 + 4lh.$$

Note that the box has no top and the above equation reflects this detail. Substituting the value of h from the volume equation into the area equation we obtain the one-dimensional function to be minimized as follows

$$f(l) = l^2 + 4l(500/l^2)$$

The expression is further simplified as

$$f(l) = l^2 + 2000/l$$

where

$$f'(l) = 2l - 2000/l^2$$

and

$$f''(l) = 2 + 4000/l^3$$

Using Newton's method and an initial guess of $l = 5\,in$, the first iteration would be:

<u>i=0</u>

$$\begin{aligned} x_1 &= x_0 - \frac{f'(x_0)}{f''(x_0)} \\ &= 5 - \frac{f'(5)}{f''(5)} \\ &= 5 - \frac{2(5) - 2000/(25)}{2 + 4000/(125)} \\ &= 7.0588 \end{aligned}$$

The second iteration would be

<u>i=1</u>

$$
\begin{aligned}
x_2 &= x_1 - \frac{f'(x_1)}{f''(x_1)} \\
&= 7.0588 - \frac{f'(7.0588)}{f''(7.0588)} \\
&= 7.0588 - \frac{2(7.0588) - 2000/(7.0588)^2}{2 + 4000/(7.0588)^3} \\
&= 9.0047
\end{aligned}
$$

Using equation the $l^2 h = 500$ and $l = 9.00$ we can determine $h = 6.17$ therefore the correct answer is (C). The minimum material used is 303.19 square feet. The optimal answer is where $l = 10.00$ and $h = 5.00$ that is obtained in 6 iterations as shown below.

Iteration	l	$f'(l)$	$f''(l)$	l est.	h	$f(l)$	volume
1	5.0000	-70.000	34.000	7.0588	10.035	333.16	500.00
2	7.0588	-26.021	13.373	9.0047	6.1664	303.19	500.00
3	9.0047	-6.6564	7.4784	9.8948	5.1069	300.03	500.00
4	9.8948	-0.6382	6.1290	9.9989	5.0011	300.00	500.00
5	9.9989	-0.0067	6.0013	10.000	5.0000	300.00	500.00
6	10.0000	0.0000	6.0000	10.000	5.0000	300.00	500.00

6. A rectangular box with a square base with no top has a surface area of 108 ft2. Find the dimensions that will maximize the volume. Conduct two iterations using an initial guess of $l = 3\ ft$

6.
   (A) Base edge length is 4.15 and height is 4.85
   (B) Base edge length is 6.15 and height is 2.85
   (C) Base edge length is 6.00and height is 3.00
   (D) Base edge length is 3.85and height is 6.15

Solution

The correct answer is (B).

This is a slightly different version of the previous problem where now we are trying to maximize volume given an area. We can relate edge length l with the height of the box h as follows:

$$A = 108 = l^2 + 4lh$$

The box with the maximum volume requires maximizing the function

$$f(l,h) = l^2 h$$

Substituting the value of h in terms of l, we obtain the following one-dimensional optimization function.

$$f(l) = l^2 \frac{(108 - l^2)}{4l}$$

$$= 27l - \frac{l^3}{4}$$

where

$$f^{'}(l) = 27 - \frac{3}{4}l^2$$

and

$$f^{''}(l) = -\frac{3}{2}l$$

Using an initial estimate of 3, and using Newton's method, the first iteration would be:

$$
\begin{aligned}
x_1 &= x_0 - \frac{f'(x_0)}{f''(x_0)} \\
&= 3 - \frac{f'(3)}{f''(3)} \\
&= 3 - \frac{27 - 0.75(9)}{-1.5(3)} \\
&= 7.5000
\end{aligned}
$$

The second iteration would be

$$
\begin{aligned}
x_2 &= x_1 - \frac{f'(x_1)}{f''(x_1)} \\
&= 7.5000 - \frac{f'(7.5000)}{f''(7.5000)} \\
&= 7.5000 - \frac{27 - 0.75(7.5000)^2}{-1.5(7.5000)} \\
&= 6.1500
\end{aligned}
$$

Using the equation $A = 108 = l^2 + 4lh$ and $l = 6.15$,we can determine $h = 2.85$ .Therefore, the correct answer is (B). The maximum volume is 107.90 square feet. The optimal answer is where $l = 6$ and $h = 3$ that is obtained in 5 iterations as shown below.

Iteration	l	$f'(l)$	$f''(l)$	l est.	h	$f(l)$	area
1	3.0000	20.250	-4.5000	7.5000	1.7250	97.031	108.00
2	7.5000	-15.188	-11.2500	6.1500	2.8527	107.90	108.00
3	6.1500	-1.3699	-9.2250	6.0018	2.9982	108.00	108.00
4	6.0018	-0.0165	-9.0027	6.0000	3.0000	108.00	108.00
5	6.0000	0.0000	-9.0000	6.0000	3.0000	108.00	108.00