Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Elliptic Partial Differential Equations
Partial Differential Equations
COMPLETE SOLUTION SET

1. In a general second order linear partial differential equation with two independent variables,

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D=0$$

where A, B, C are functions of x and y, and D is a function of x, y, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, then the PDE is elliptic if

(A) $B^2-4AC<0$
(B) $B^2-4AC>0$
(C) $B^2-4AC=0$
(D) $B^2-4AC\neq 0$

Solution
The correct answer is (A).

In a general second order linear partial differential equation with two independent variables,

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D=0$$

where A, B, C are functions of x and y, and D is a function of x, y, $\frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial y}$.

If $B^2-4AC<0$, the second order linear partial differential equation is elliptic.

2. The region in which the following equation

$$x^3 \frac{\partial^2 u}{\partial x^2} + 27\frac{\partial^2 u}{\partial y^2} + 3\frac{\partial^2 u}{\partial x \partial y} + 5u = 0$$

acts as an elliptic equation is

(A) $x > \left(\frac{1}{12}\right)^{1/3}$

(B) $x < \left(\frac{1}{12}\right)^{1/3}$

(C) for all values of x

(D) $x = \left(\frac{1}{12}\right)^{1/3}$

Solution

The correct answer is (A).

A general partial differential equation with two independent variables is of the from

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial xy} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

where A, B, and C are functions of x and y and is a function of x, y, u and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$.

For this equation to be elliptic,

$$B^2 - 4AC < 0.$$

In the above question,

$$A = x^3, B = 3, C = 27, D = 5u,$$

giving

$$B^2 - 4AC < 0$$

$$(3)^2 - 4(x^3)(27) < 0$$

$$9 - 108x^3 < 0$$

$$108x^3 > 9$$

$$x^3 > \frac{9}{108}$$

$$x^3 > \frac{1}{12}$$

$$x > \left(\frac{1}{12}\right)^{1/3}$$

3. The finite difference approximation of $\frac{\partial^2 u}{\partial x^2}$ in the elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

at (x, y) can be approximated as

(A) $\frac{\partial^2 u}{\partial x^2} \cong \frac{u(x+\Delta x, y) - 2u(x, y) + u(x-\Delta x, y)}{(\Delta x)^2}$

(B) $\frac{\partial^2 u}{\partial x^2} \cong \frac{u(x+\Delta x, y) - u(x, y) + u(x-\Delta x, y)}{(\Delta x)^2}$

(C) $\frac{\partial^2 u}{\partial x^2} \cong \frac{u(x, y+\Delta y) - 2u(x, y) + u(x, y-\Delta y)}{(\Delta x)^2}$

(D) $\frac{\partial^2 u}{\partial x^2} \cong \frac{u(x+\Delta x, y) - u(x-\Delta x, y)}{2\Delta x}$

Solution

The correct answer is (A).

The Taylor series for a two-dimensional function $u(x, y)$ is given by

$$u(x+\Delta x, y+\Delta y) = u(x, y) + u_x(x, y)\Delta x + u_y(x, y)\Delta y + \frac{1}{2}u_{xx}(x, y)(\Delta x)^2$$
$$+ \frac{1}{2}u_{yy}(x, y)(\Delta y)^2 + u_{xy}(x, y)(\Delta x)(\Delta y) + \ldots$$

For $u(x+\Delta x, y)$, that is $(\Delta y) = 0$, the Taylor series would reduce to

$$u(x+\Delta x, y) = u(x, y) + u_x(x, y)\Delta x + \frac{1}{2}u_{xx}(x, y)(\Delta x)^2 + \ldots\ldots\ldots \quad (1)$$

Then

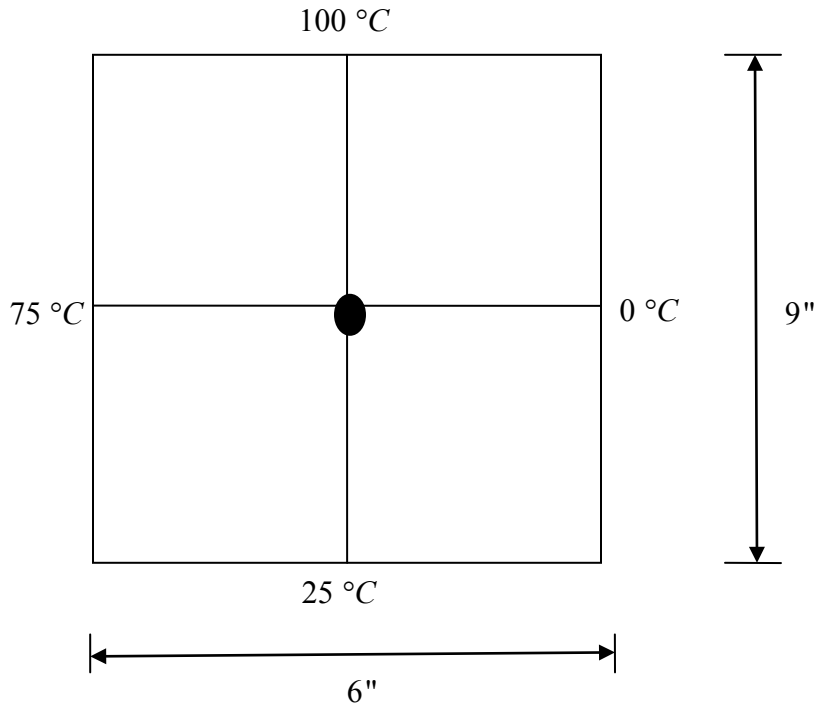
$$u(x-\Delta x, y) = u(x, y) - u_x(x, y)\Delta x + \frac{1}{2}u_{xx}(x, y)(\Delta x)^2 + \ldots\ldots\ldots \quad (2)$$

Adding Equation (1) and Equation (2) gives

$$u(x+\Delta x, y) + u(x-\Delta x, y) = 2u(x, y) + u_{xx}(x, y)(\Delta x)^2 + \ldots\ldots\ldots$$

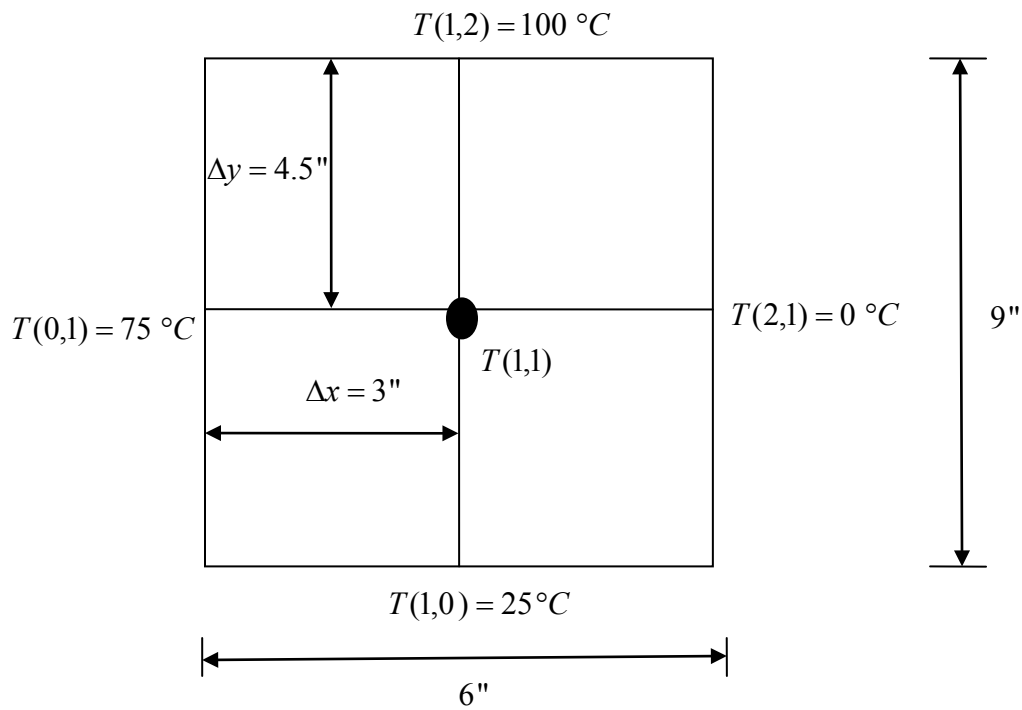
$$u_{xx}(x, y) \cong \frac{u(x+\Delta x, y) - 2u(x, y) + u(x-\Delta x, y)}{(\Delta x)^2}$$

4. Find the temperature at the interior node given in the following figure using the direct method



(A) 45.19 °C
(B) 48.64 °C
(C) 50.00 °C
(D) 56.79 °C

Solution
The correct answer is (A).



From Equation (9) of chapter 10.3

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

putting $i = 1$ and $j = 1$, we have

$$\frac{T_{2,1} - 2T_{1,1} + T_{0,1}}{(\Delta x)^2} + \frac{T_{1,2} - 2T_{1,1} + T_{1,0}}{(\Delta y)^2} = 0$$

$$\frac{0 - 2T_{1,1} + 75}{(3)^2} + \frac{100 - 2T_{1,1} + 25}{(4.5)^2} = 0$$

$$\frac{75 - 2T_{1,1}}{9} + \frac{125 - 2T_{1,1}}{20.25} = 0$$

$$\frac{75}{9} + \frac{125}{20.25} - 2T_{1,1}\left(\frac{1}{9} + \frac{1}{20.25}\right) = 0$$

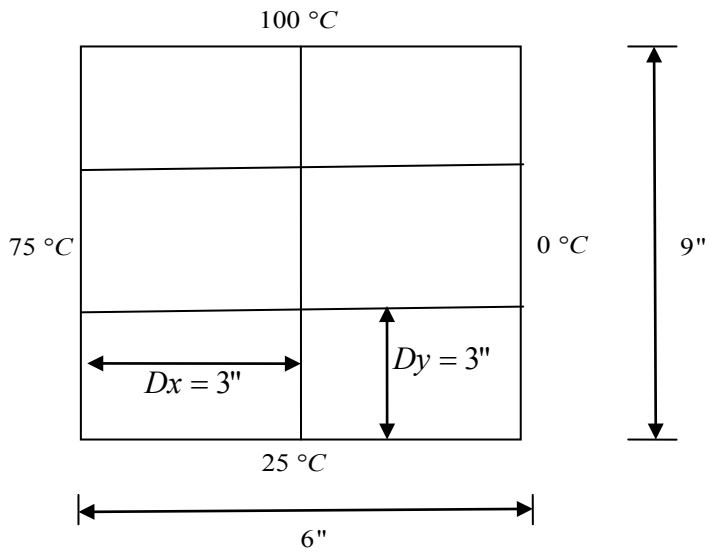
$$8.333 + 6.173 - 2T_{1,1}(0.1605) = 0$$

$$14.51 - 2T_{1,1}(0.1605) = 0$$

$$2T_{1,1}(0.1605) = 14.51$$

$$T_{1,1} = 45.19\ °C$$

5. Find the temperature at the interior node given in the following figure

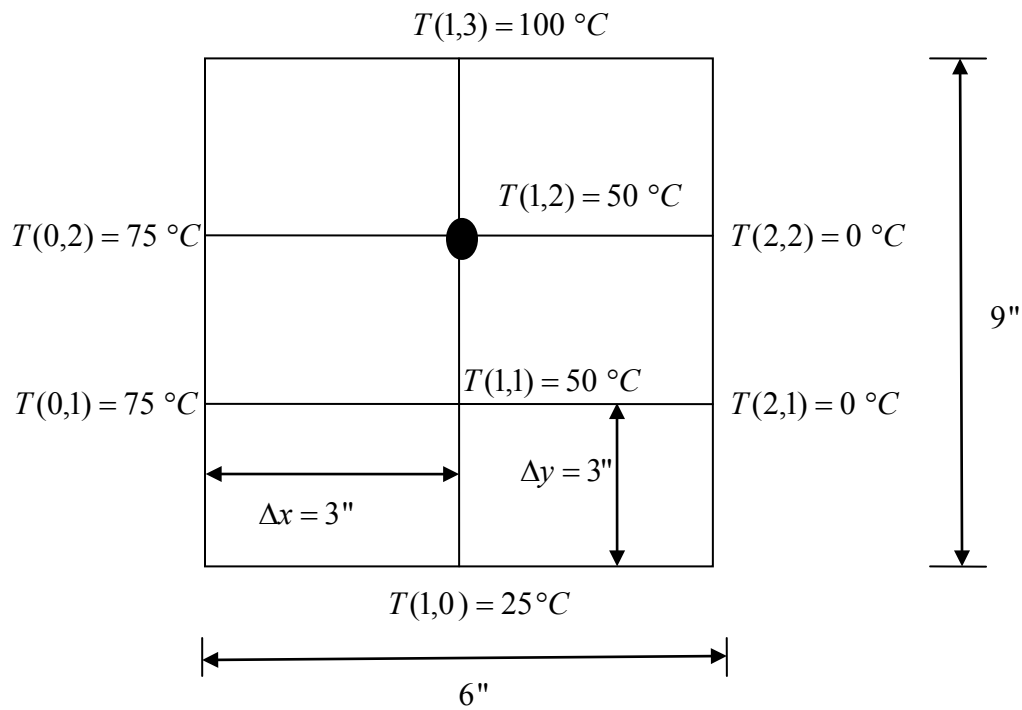


Using the Lieberman method and a relaxation factor of 1.2, the temperature at $x = 3, y = 6$ estimated after 2 iterations is (use the temperature of interior nodes as $50\ °C$ for the initial guess)

(A) $52.36\,°C$
(B) $53.57\,°C$
(C) $56.20\,°C$
(D) $58.64\,°C$

Solution
The correct answer is (B).

To solve this problem we will choose a grid of $\Delta x = \Delta y = 3"$.



From the boundary conditions

$$\left.\begin{aligned} T_{i,0} &= 25\ °C; i = 1 \\ T_{i,3} &= 100\ °C; i = 1 \\ T_{0,j} &= 75\ °C; j = 1,2 \\ T_{2,j} &= 0\ °C; j = 1,2 \end{aligned}\right\} \tag{1}$$

Now to get the temperature at the interior nodes, we have to write Equation (11) (from Chapter 10.03) for all the combinations of i and j, $i = 1,2,3$ and $j = 1,2$. After getting the temperature from Equation (11), we have to use Equation (12) (from Chapter 10.03) to apply the over relaxation method.

<u>Iteration 1</u>

For iteration 1, we start with all of the interior nodes having an initial temperature of $50°C$.

<u>*i*=1 and *j*=1</u>

$$\begin{aligned} T_{1,1} &= \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4} \\ &= \frac{0 + 75 + 50 + 25}{4} \\ &= 37.50°C \end{aligned}$$

$$\begin{aligned} T_{1,1}^{relaxed} &= \lambda T_{1,1}^{new} + (1-\lambda) T_{1,1}^{old} \\ &= 1.2(37.50) + (1-1.2)50 \\ &= 35.00°C \end{aligned}$$

<u>*i*=1 and *j*=2</u>

$$T_{1,2} = \frac{T_{2,2} + T_{0,2} + T_{1,3} + T_{1,1}}{4}$$
$$= \frac{0 + 75 + 100 + 35}{4}$$
$$= 52.50°C$$
$$T_{1,2}^{relaxed} = \lambda T_{1,2}^{new} + (1-\lambda)T_{1,2}^{old}$$
$$= 1.2(52.50) + (1-1.2)50.00$$
$$= 53.00°C$$

<u>Iteration 2</u>

For iteration 2, we take the temperatures from iteration 1.

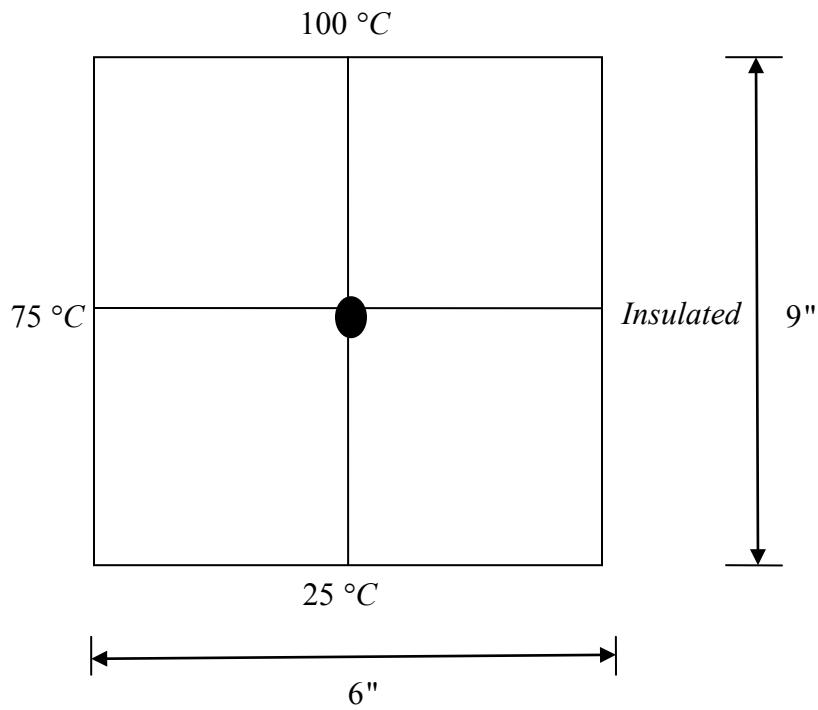
<u>*i*=1 and *j*=1</u>

$$T_{1,1} = \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4}$$
$$= \frac{0 + 75 + 53 + 25}{4}$$
$$= 38.25°C$$
$$T_{1,1}^{relaxed} = \lambda T_{1,1}^{new} + (1-\lambda)T_{1,1}^{old}$$
$$= 1.2(38.25) + (1-1.2)35$$
$$= 38.90°C$$

<u>*i*=1 and *j*=2</u>

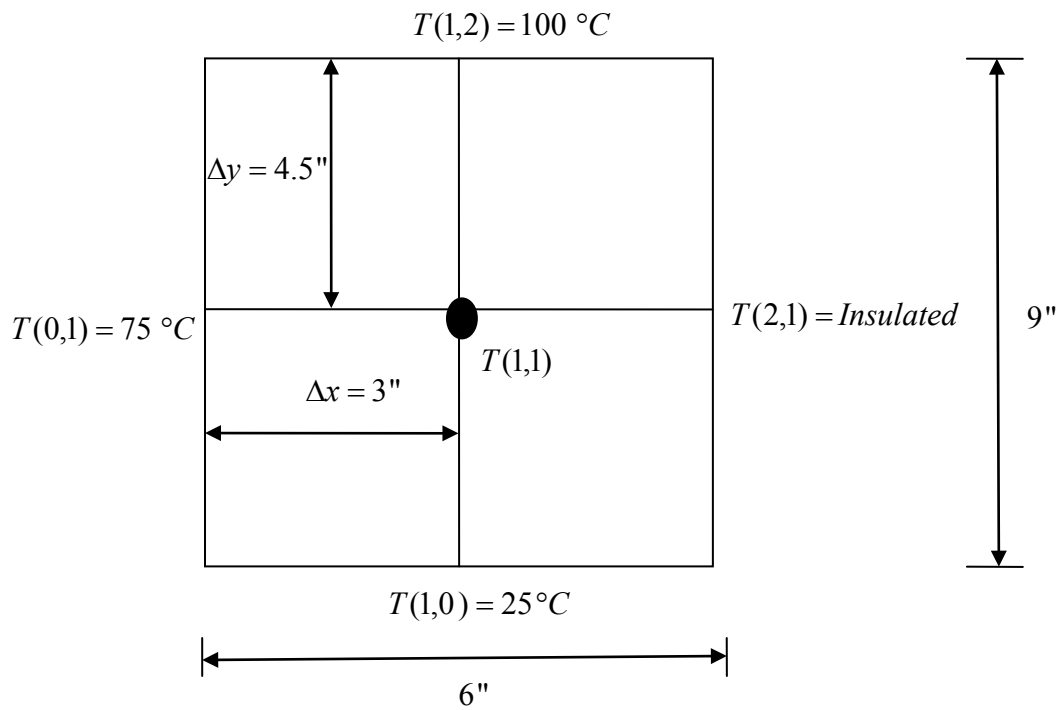
$$T_{1,2} = \frac{T_{2,2} + T_{0,2} + T_{1,3} + T_{1,1}}{4}$$
$$= \frac{0 + 75 + 100 + 38.90}{4}$$
$$= 53.48°C$$
$$T_{1,2}^{relaxed} = \lambda T_{1,2}^{new} + (1-\lambda)T_{1,2}^{old}$$
$$= 1.2(53.48) + (1-1.2)53.00$$
$$= 53.57°C$$

6. Find the steady-state temperature at the interior node as given in the following figure



(A) 53.57°C
(B) 66.40°C
(C) 68.20°C
(D) 69.59°C

Solution
The correct answer is (C).



From Equation (9) of chapter 10.3

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

for $i = 1$ and $j = 1$, we have

$$\frac{T_{2,1} - 2T_{1,1} + T_{0,1}}{(\Delta x)^2} + \frac{T_{1,2} - 2T_{1,1} + T_{1,0}}{(\Delta y)^2} = 0$$

$$\frac{T_{2,1} - 2T_{1,1} + 75}{(3)^2} + \frac{100 - 2T_{1,1} + 25}{(4.5)^2} = 0$$

$$\frac{75 - 2T_{1,1} + T_{2,1}}{9} + \frac{125 - 2T_{1,1}}{20.25} = 0$$

$$\frac{75}{9} + \frac{125}{20.25} - 2T_{1,1}\left(\frac{1}{9} + \frac{1}{20.25}\right) + \frac{T_{2,1}}{9} = 0$$

$$8.333 + 6.173 - 2T_{1,1}(0.1605) + \frac{T_{2,1}}{9} = 0$$

$$14.51 - 2T_{1,1}(0.1605) + \frac{T_{2,1}}{9} = 0$$

$$-0.3210T_{1,1} + 0.1111T_{2,1} = -14.51 \qquad \text{(E6.1)}$$

for $i = 2$ and $j = 1$, we have

$$\frac{T_{3,1}-2T_{2,1}+T_{1,1}}{(\Delta x)^2}+\frac{T_{2,2}-2T_{2,1}+T_{2,0}}{(\Delta y)^2}=0 \tag{E6.2}$$

However, the node $(3,1)$ is not inside the plate. The derivative boundary condition needs to be used to account for these additional unknown nodal temperatures on the right edge. This is done by approximating the derivative at the edge node $(2,1)$ as

$$\left.\frac{\partial T}{\partial x}\right|_{2,1} \cong \frac{T_{3,1}-T_{1,1}}{2(\Delta x)}$$

giving

$$T_{3,1}=T_{1,1}+2(\Delta x)\left.\frac{\partial T}{\partial x}\right|_{2,1} \tag{E6.3}$$

Substituting (E.6.3) in (E6.2) we have

$$\frac{T_{1,1}+2(\Delta x)\left.\frac{\partial T}{\partial x}\right|_{2,1}-2T_{2,1}+T_{1,1}}{(\Delta x)^2}+\frac{T_{2,2}-2T_{2,1}+T_{2,0}}{(\Delta y)^2}=0 \tag{E6.4}$$

But for the insulated edge

$$\left.\frac{\partial T}{\partial x}\right|_{2,1}=0 \tag{E6.5}$$

Substituting (E6.5) in (E6.4) we have

$$\frac{T_{1,1}-2T_{2,1}+T_{1,1}}{(\Delta x)^2}+\frac{T_{2,2}-2T_{2,1}+T_{2,0}}{(\Delta y)^2}=0$$

$$\frac{2T_{1,1}-2T_{2,1}}{(\Delta x)^2}+\frac{T_{2,2}-2T_{2,1}+T_{2,0}}{(\Delta y)^2}=0$$

$$\frac{2T_{1,1}-2T_{2,1}}{(3)^2}+\frac{100-2T_{2,1}+25}{(4.5)^2}=0$$

$$\frac{2T_{1,1}-2T_{2,1}}{9}+\frac{125-2T_{2,1}}{20.25}=0$$

$$0.2222T_{1,1}-0.3210T_{2,1}=-6.172 \tag{E6.6}$$

Solving Equations (E6.1) and (E6.6)

$$-0.3210T_{1,1}+0.1111T_{2,1}=-145.51$$

$$0.2222T_{1,1}-0.3210T_{2,1}=-6.172$$

$$T_{1,1}=68.20\ °C$$