Newton's Divided Difference Polynomial Method of Interpolation

Chemical Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

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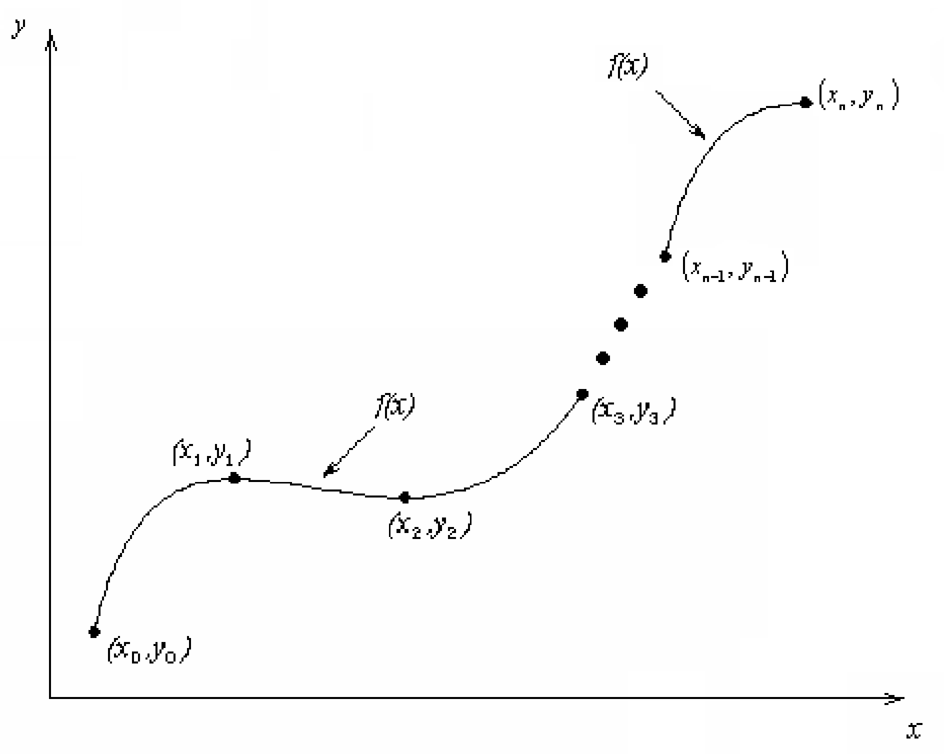
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

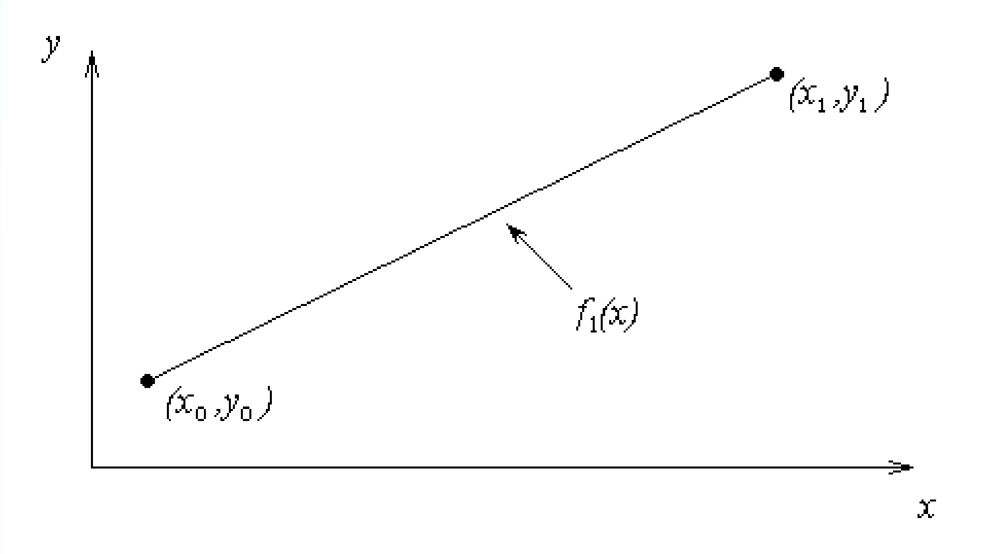
Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Newton's Divided Difference Method

<u>Linear interpolation</u>: Given $(x_0, y_0), (x_1, y_1),$ pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$



where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Example

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at 61˚C. The specific heat of water is given as a function of time in Table 1. Use Newton's divided difference method with a first order and then a second order polynomial to determine the value of the specific heat at T = 61˚C.

Table 1 Specific heat of water as a function of temperature.

Temperature, $T\ (°\mathrm{C})$	Specific heat, $C_p\ \left(\frac{\mathrm{J}}{\mathrm{kg}-°\mathrm{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

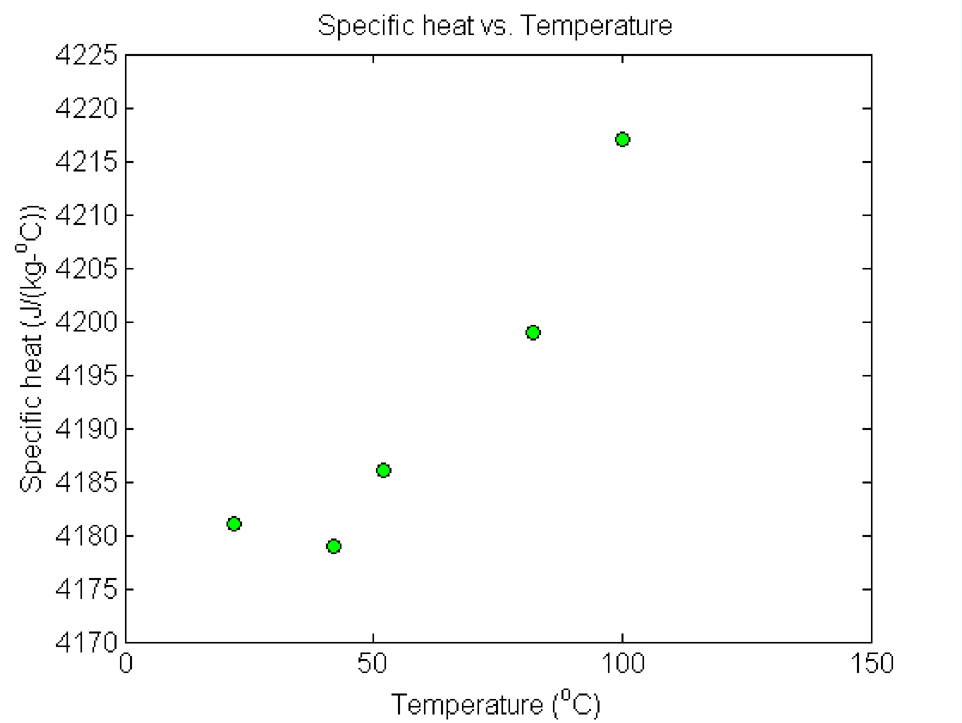


Figure 2 Specific heat of water vs. temperature.

Linear Interpolation

$$C_p(T) = b_0 + b_1(T - T_0)$$

$$T_0 = 52,\ C_p(T_0) = 4186$$

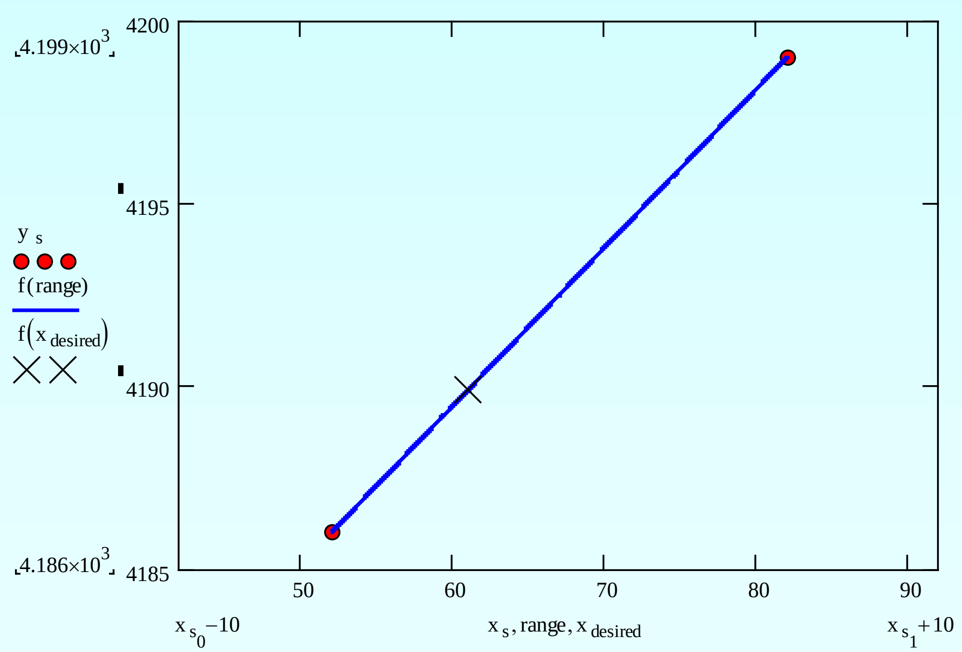
$$T_1 = 82,\ C_p(T_1) = 4199$$

$$b_0 = C_p(T_0) = 4186$$

$$b_1 = \frac{C_p(T_1) - C_p(T_0)}{T_1 - T_0}$$

$$= \frac{4199 - 4186}{82 - 52}$$

$$= 0.43333$$



Linear Interpolation (contd)

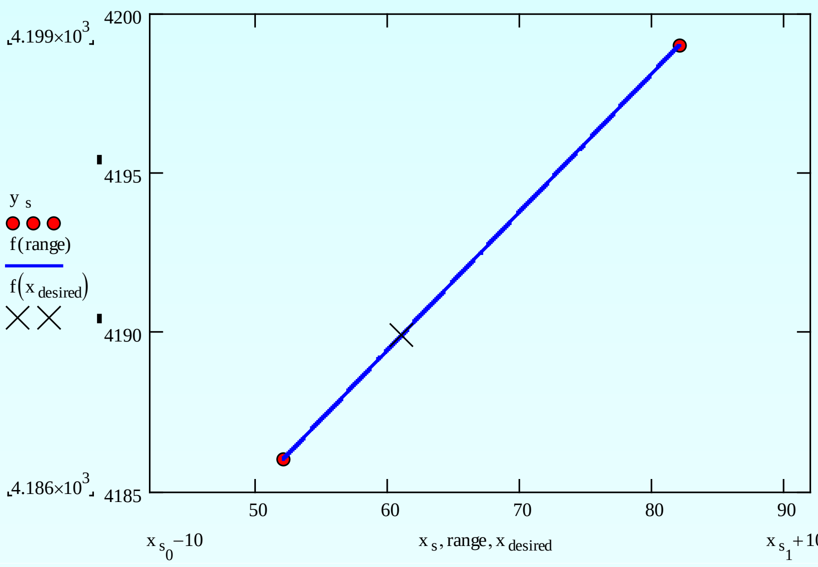
$$C_p(T) = b_0 + b_1(T - T_0)$$

$$= 4186 + 0.43333(T - 52), \quad 52 \le T \le 82$$

At $T = 61$

$$C_p(61) = 4186 + 0.43333(61 - 52)$$

$$= 4189.9 \frac{J}{kg - {}^\circ C}$$



Quadratic Interpolation

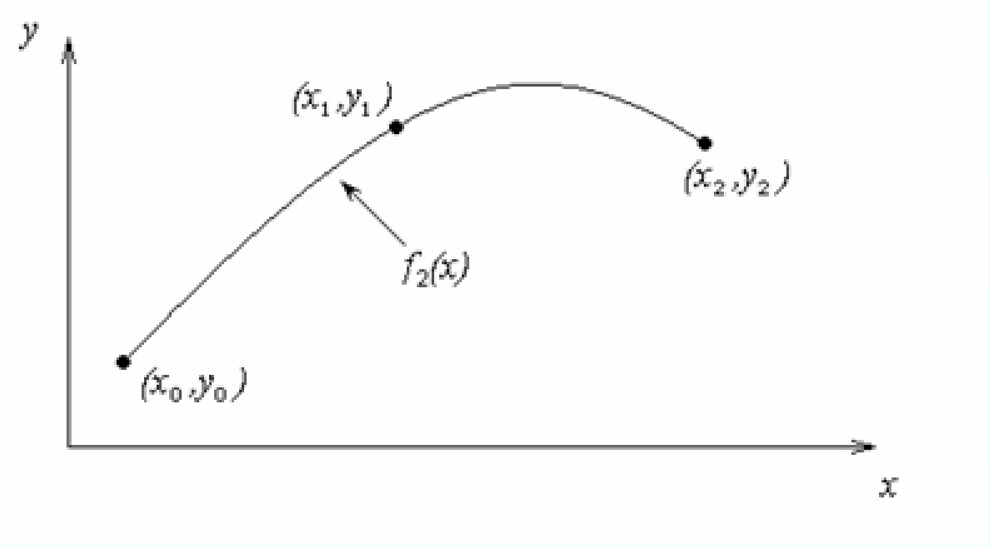
Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



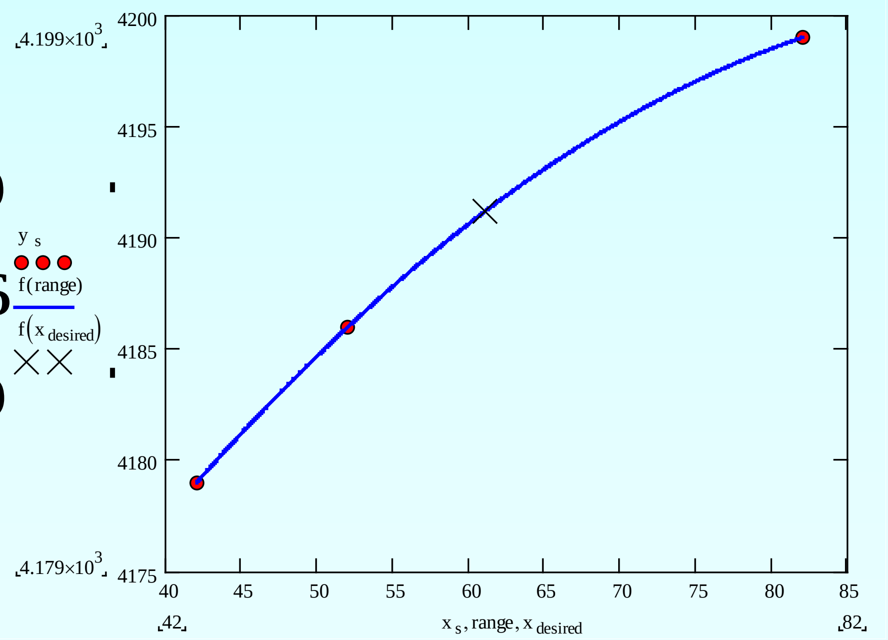
Quadratic Interpolation (contd)

$$C_p(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1)$$

$$T_0 = 42,\ C_p(T_0) = 4179$$

$$T_1 = 52,\ C_p(T_1) = 4186$$

$$T_2 = 82,\ C_p(T_2) = 4199$$



Quadratic Interpolation (contd)

$$b_0 = C_p(T_0) = 4179$$

$$b_1 = \frac{C_p(T_1) - C_p(T_0)}{T_1 - T_0} = \frac{4186 - 4179}{52 - 42} = 0.7$$

$$b_2 = \frac{\dfrac{C_p(T_2) - C_p(T_1)}{T_2 - T_1} - \dfrac{C_p(T_1) - C_p(T_0)}{T_1 - T_0}}{T_2 - T_0} = \frac{\dfrac{4199 - 4186}{82 - 52} - \dfrac{4186 - 4179}{52 - 42}}{82 - 42}$$

$$= \frac{0.43333 - 0.7}{40}$$

$$= -6.6667 \times 10^{-3}$$

Quadratic Interpolation (contd)

$$C_p(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1)$$

$$= 4179 + 0.7(T - 42) - 6.6667\times 10^{-3}(T - 42)(T - 52), \qquad 42 \le T \le 82$$

At $T = 61$,

$$C_p(61) = 4179 + 0.7(61 - 42) + 6.6667\times 10^{-3}(61 - 42)(61 - 52)$$

$$= 4191.2 \frac{J}{kg - ^\circ C}$$

The absolute relative approximate error $\left|\in_a\right|$ obtained between the results from the first and second order polynomial is

$$\left|\in_a\right| = \left|\frac{4191.2 - 4189.9}{4191.2}\right| \times 100$$

$$= 0.030063\%$$

General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

General Form

Given $(n+1)$ data points, $(x_0, y_0),(x_1, y_1),......,(x_{n-1}, y_{n-1}),(x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + ... + b_n(x - x_0)(x - x_1)...(x - x_{n-1})$$

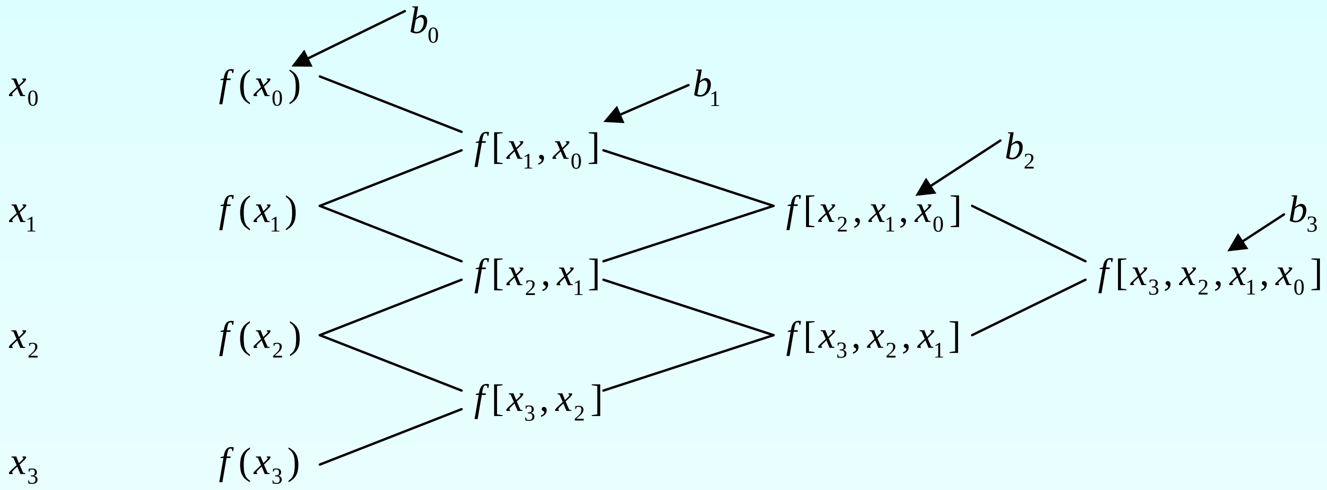
where

$$b_0 = f[x_0]$$
$$b_1 = f[x_1, x_0]$$
$$b_2 = f[x_2, x_1, x_0]$$
$$\vdots$$
$$b_{n-1} = f[x_{n-1}, x_{n-2}, ..., x_0]$$
$$b_n = f[x_n, x_{n-1}, ..., x_0]$$

General form

The third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3), is

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$
$$+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$



Example

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at 61˚C. The specific heat of water is given as a function of time in Table 1. Use Newton's divided difference method with a third order polynomial to determine the value of the specific heat at T = 61˚C.

Table 1 Specific heat of water as a function of temperature.

Temperature, $T\ (°C)$	Specific heat, $C_p\left(\frac{J}{kg-°C}\right)$
22	4181
42	4179
52	4186
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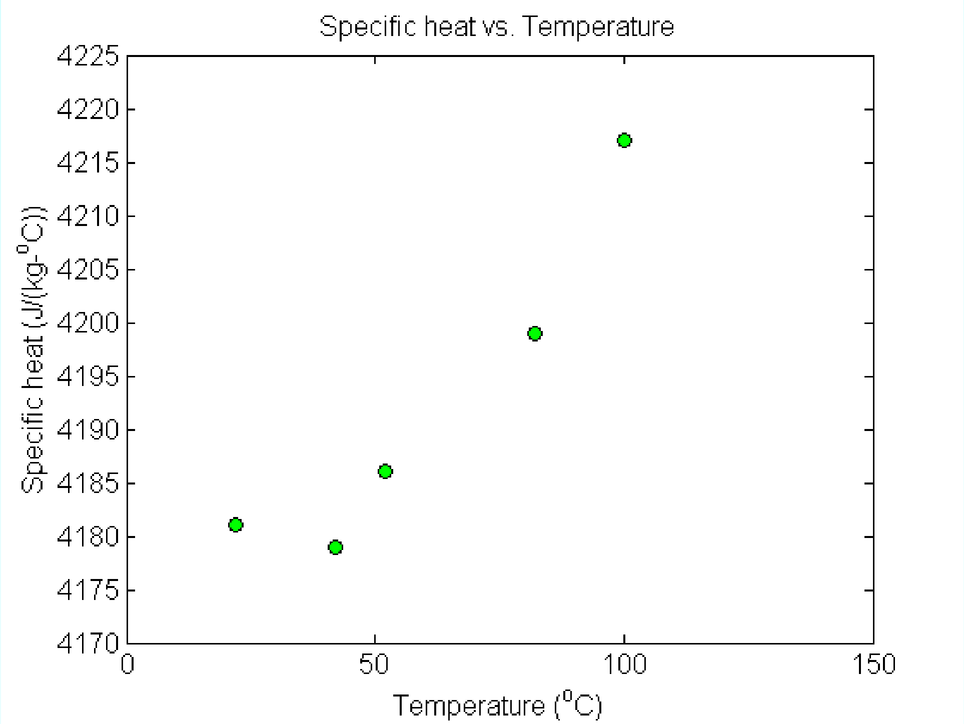

Figure 2 Specific heat of water vs. temperature.

Example

The specific heat profile is chosen as

$$C_p(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1) + b_3(T - T_0)(T - T_1)(T - T_2)$$

We need to choose four data points that are closest to $T = 61°C$.

$$T_0 = 42, \quad C_p(T_0) = 4179$$

$$T_1 = 52, \quad C_p(T_1) = 4186$$

$$T_2 = 82, \quad C_p(T_2) = 4199$$

$$T_3 = 100, \quad C_p(T_3) = 4217$$

Example

$T_0 = 42,$	4179 (b_0)			
		0.7 (b_1)		
$T_1 = 52,$	4186		-6.6667×10^{-3} (b_2)	
		0.43333		3.1849×10^{-4} (b_3)
$T_2 = 82,$	4199		0.011806	
		1		
$T_3 = 100,$	4217			

The values of the constants are found to be

$b_0 = 4179$ $\quad b_1 = 0.7$ $\quad b_2 = -6.6667\times10^{-3}$ $\quad b_3 = 3.1849\times10^{-4}$

Example

$$C_p(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1) + b_3(T - T_0)(T - T_1)(T - T_2)$$

$$= 4179 + 0.7(T - 42) - 6.6667 \times 10^{-3}(T - 42)(T - 52)$$

$$+ 3.1849 \times 10^{-4}(T - 42)(T - 52)(T - 82) \qquad 42 \le T \le 100$$

At $T = 61$,

$$C_p(61) = 4179 + 0.7(61 - 42) - 6.6667 \times 10^{-3}(61 - 42)(61 - 52)$$

$$+ 3.1849 \times 10^{-4}(61 - 42)(61 - 52)(61 - 82)$$

$$= 4190.0 \frac{J}{kg - {}^\circ C}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a| = \left|\frac{4190.0 - 4191.2}{4190.0}\right| \times 100$$

$$= 0.027295\,\%$$

Comparison Table

Order of Polynomial	1	2	3
$C_p(T)\ \dfrac{J}{kg-^\circ C}$	4189.9	4191.2	4190.0
Absolute Relative Approximate Error	----------	0.030063%	0.027295%

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/newton_divided_difference_method.html

THE END

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