Chapter 08.04
Runge-Kutta 4th Order Method for Ordinary Differential Equations

After reading this chapter, you should be able to
1. *develop Runge-Kutta 4th order method for solving ordinary differential equations,*
2. *find the effect size of step size has on the solution,*
3. *know the formulas for other versions of the Runge-Kutta 4th order method*

What is the Runge-Kutta 4th order method?

Runge-Kutta 4th order method is a numerical technique used to solve ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

So only first order ordinary differential equations can be solved by using the Runge-Kutta 4th order method. In other sections, we have discussed how Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \ y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example 2

Rewrite

$$e^y \frac{dy}{dx} + x^2 y^2 = 2\sin(3x), \ y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \ y(0) = y_0 \text{ form.}$$

Solution

$$e^y \frac{dy}{dx} + x^2 y^2 = 2\sin(3x), \ y(0) = 5$$

$$\frac{dy}{dx} = \frac{2\sin(3x) - x^2 y^2}{e^y}, \ y(0) = 5$$

In this case

$$f(x, y) = \frac{2\sin(3x) - x^2 y^2}{e^y}$$

The Runge-Kutta 4th order method is based on the following

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2 + a_3 k_3 + a_4 k_4)h \tag{1}$$

where knowing the value of $y = y_i$ at x_i, we can find the value of $y = y_{i+1}$ at x_{i+1}, and

$$h = x_{i+1} - x_i$$

Equation (1) is equated to the first five terms of Taylor series

$$y_{i+1} = y_i + \frac{dy}{dx}\bigg|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!} \frac{d^2 y}{dx^2}\bigg|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!} \frac{d^3 y}{dx^3}\bigg|_{x_i, y_i} (x_{i+1} - x_i)^3$$

$$+ \frac{1}{4!} \frac{d^4 y}{dx^4}\bigg|_{x_i, y_i} (x_{i+1} - x_i)^4 \tag{2}$$

Knowing that $\dfrac{dy}{dx} = f(x, y)$ and $x_{i+1} - x_i = h$

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2 + \frac{1}{3!}f''(x_i, y_i)h^3 + \frac{1}{4!}f'''(x_i, y_i)h^4 \tag{3}$$

Based on equating Equation (2) and Equation (3), one of the popular solutions used is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \tag{4}$$

$$k_1 = f(x_i, y_i) \tag{5a}$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) \tag{5b}$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right) \tag{5c}$$

$$k_4 = f(x_i + h, y_i + k_3 h) \tag{5d}$$

Example 3

The concentration of salt x in a home made soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5x$$

At the initial time, $t = 0$, the salt concentration in the tank is 50 g/L. Using Runge-Kutta 4th order method and a step size of, $h = 1.5 \text{ min}$, what is the salt concentration after 3 minutes?

Solution

$$\frac{dx}{dt} = 37.5 - 3.5x$$

$$f(t,x) = 37.5 - 3.5x$$

$$x_{i+1} = x_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

For $i = 0$, $t_0 = 0$, $x_0 = 50$

$$k_1 = f(t_0, x_0)$$
$$= f(0,50)$$
$$= 37.5 - 3.5(50)$$
$$= -137.5$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, x_0 + \frac{1}{2}k_1 h\right)$$
$$= f\left(0 + \frac{1}{2}1.5, 50 + \frac{1}{2}(-137.5)1.5\right)$$
$$= f(0.75, -53.125)$$
$$= 37.5 - 3.5(-53.125)$$
$$= 223.44$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, x_0 + \frac{1}{2}k_2 h\right)$$
$$= f\left(0 + \frac{1}{2}1.5, 50 + \frac{1}{2}(223.44)1.5\right)$$
$$= f(0.75, 217.58)$$
$$= 37.5 - 3.5(217.58)$$
$$= -724.02$$

$$k_4 = f(t_0 + h, x_0 + k_3 h)$$
$$= f(0 + 1.5, 50 + (-724.03)1.5)$$
$$= f(1.5, -1036.0)$$
$$= 37.5 - 3.5(-1036.0)$$
$$= 3663.6$$

$$x_1 = x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 50 + \frac{1}{6}(-137.5 + 2(223.44) + 2(-724.02) + (3663.6))1.5$$

$$= 50 + \frac{1}{6}(2525.0)1.5$$

$$= 681.24 \text{ g/L}$$

x_1 is the approximate concentration of salt at

$$t = t_1 = t_0 + h = 0 + 1.5 = 1.5$$

$$x(1.5) \approx x_1 = 681.24 \text{ g/L}$$

For $i = 1$, $t_1 = 1.5$, $x_1 = 681.24$

$$k_1 = f(t_1, x_1)$$

$$= f(1.5, 681.24)$$

$$= 37.5 - 3.5(681.24)$$

$$= -2346.8$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, x_1 + \frac{1}{2}k_1 h\right)$$

$$= f\left(1.5 + \frac{1}{2}1.5, 681.24 + \frac{1}{2}(-2346.8)1.5\right)$$

$$= f(2.25, -1078.9)$$

$$= 37.5 - 3.5(-1078.9)$$

$$= 3813.6$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, x_1 + \frac{1}{2}k_2 h\right)$$

$$= f\left(1.5 + \frac{1}{2}1.5, 681.24 + \frac{1}{2}(3813.6)1.5\right)$$

$$= f(2.25, 3541.4)$$

$$= 37.5 - 3.5(3541.4)$$

$$= -12358$$

$$k_4 = f(t_1 + h, x_1 + k_3 h)$$

$$= f(1.5 + 1.5, 681.24 + (-12358)1.5)$$

$$= f(3, -17855)$$

$$= 37.5 - 3.5(-17855)$$

$$= 62530$$

$$x_2 = x_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 681.24 + \frac{1}{6}(-2346.8 + 2(3813.6) + 2(-12358) + 62530)1.5$$

$$= 681.24 + \frac{1}{6}(43096)1.5$$

$$= 11455 \text{ g/L}$$

x_2 is the approximate concentration of salt at

$$t_2 = t_1 + h = 1.5 + 1.5 = 3 \text{ min}$$
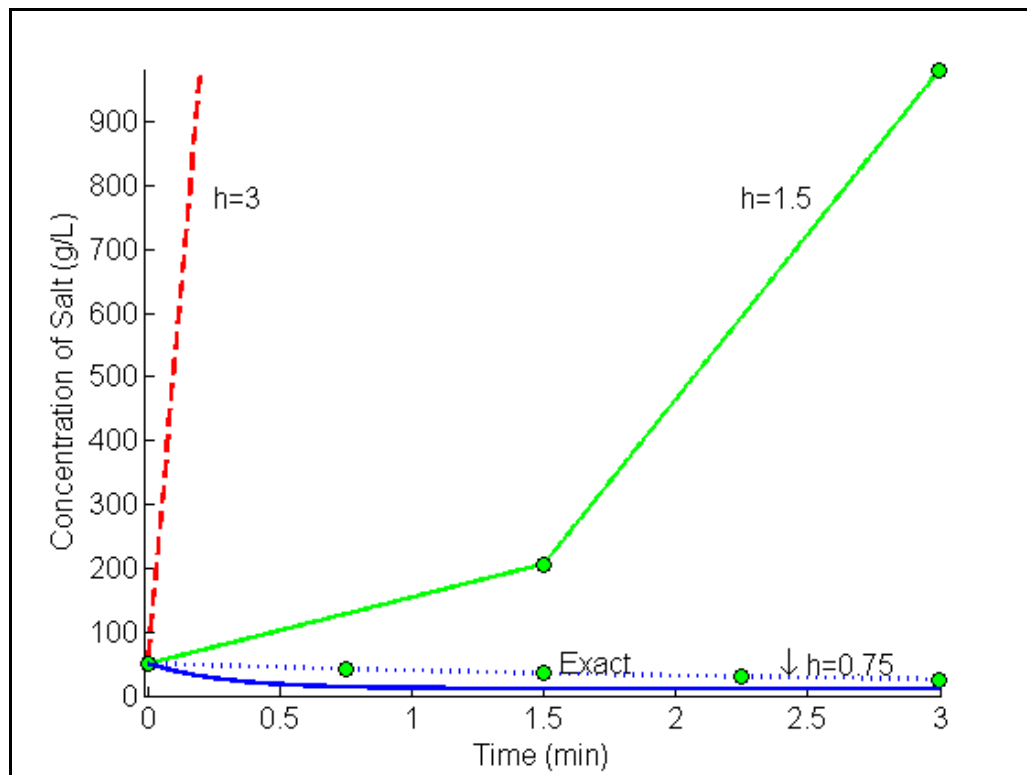
$$x(3) \approx x_2 = 11455 \text{ g/L}$$

The exact solution of the ordinary differential equation is given by

$$x(t) = 10.714 + 39.286e^{-3.5t}$$

The solution to this nonlinear equation at $t = 3$ min is

$$x(3) = 10.715 \text{ g/L}$$

Figure 1 compares the exact solution with the numerical solution using Runge-Kutta 4th order method using different step sizes.
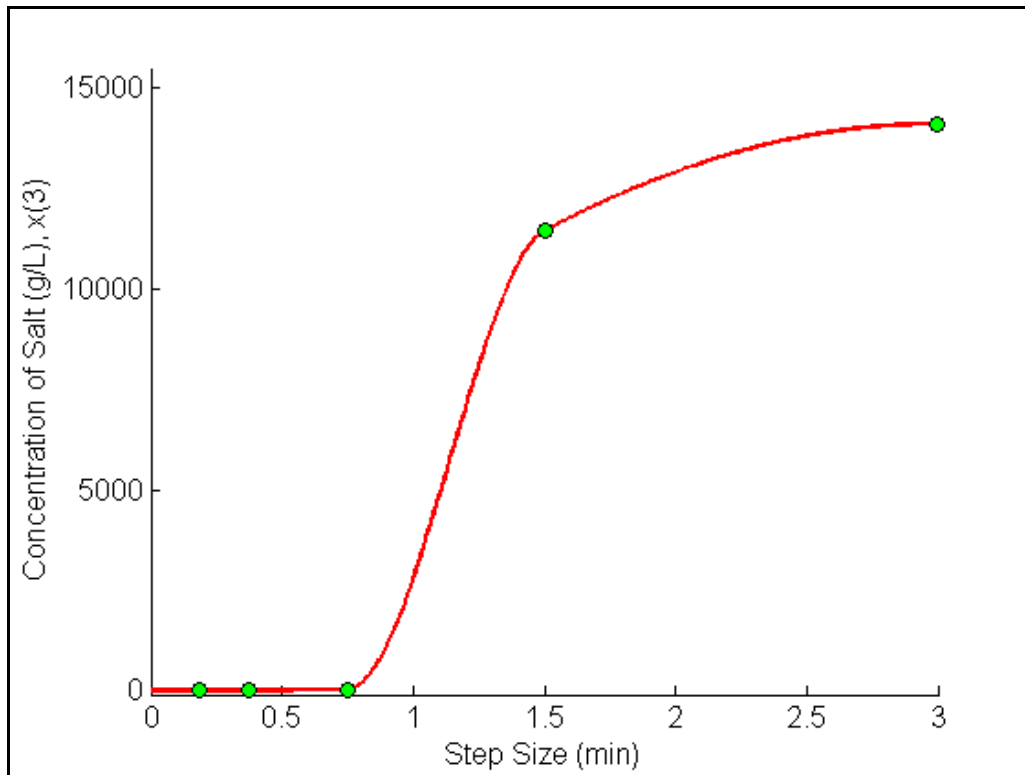

Figure 1 Comparison of Runge-Kutta 4th order method with exact solution   for different step sizes.

Table 1 and Figure 2 show the effect of step size on the value of the calculated temperature at $t = 3$ min.

Table 1 Value of concentration of salt at 3 minutes for different step sizes.

Step size, h	$x(3)$	E_t	$|\in_t|\%$
3	14120	−14109	131680
1.5	11455	−11444	106800
0.75	25.559	−14.843	138.53
0.375	10.717	−0.0014969	0.013969
0.1875	10.715	−0.00031657	0.0029544

Figure 2 Effect of step size in Runge-Kutta 4th order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method) and Runge-Kutta 4th order method.
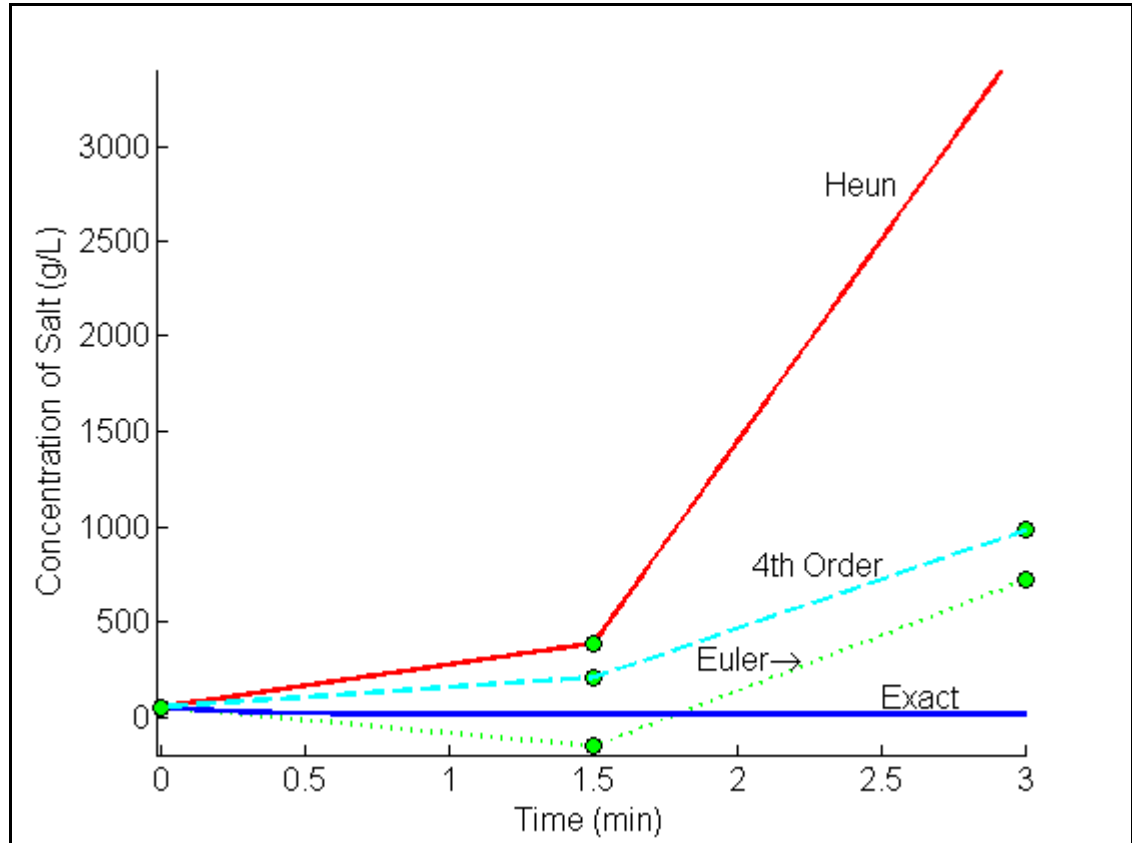
Figure 3 Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Runge-Kutta 4th order method
Summary	Textbook notes on the Runge-Kutta 4th order method for solving ordinary differential equations.
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Authors	Autar Kaw
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