Newton-Raphson Method

Civil Engineering Majors

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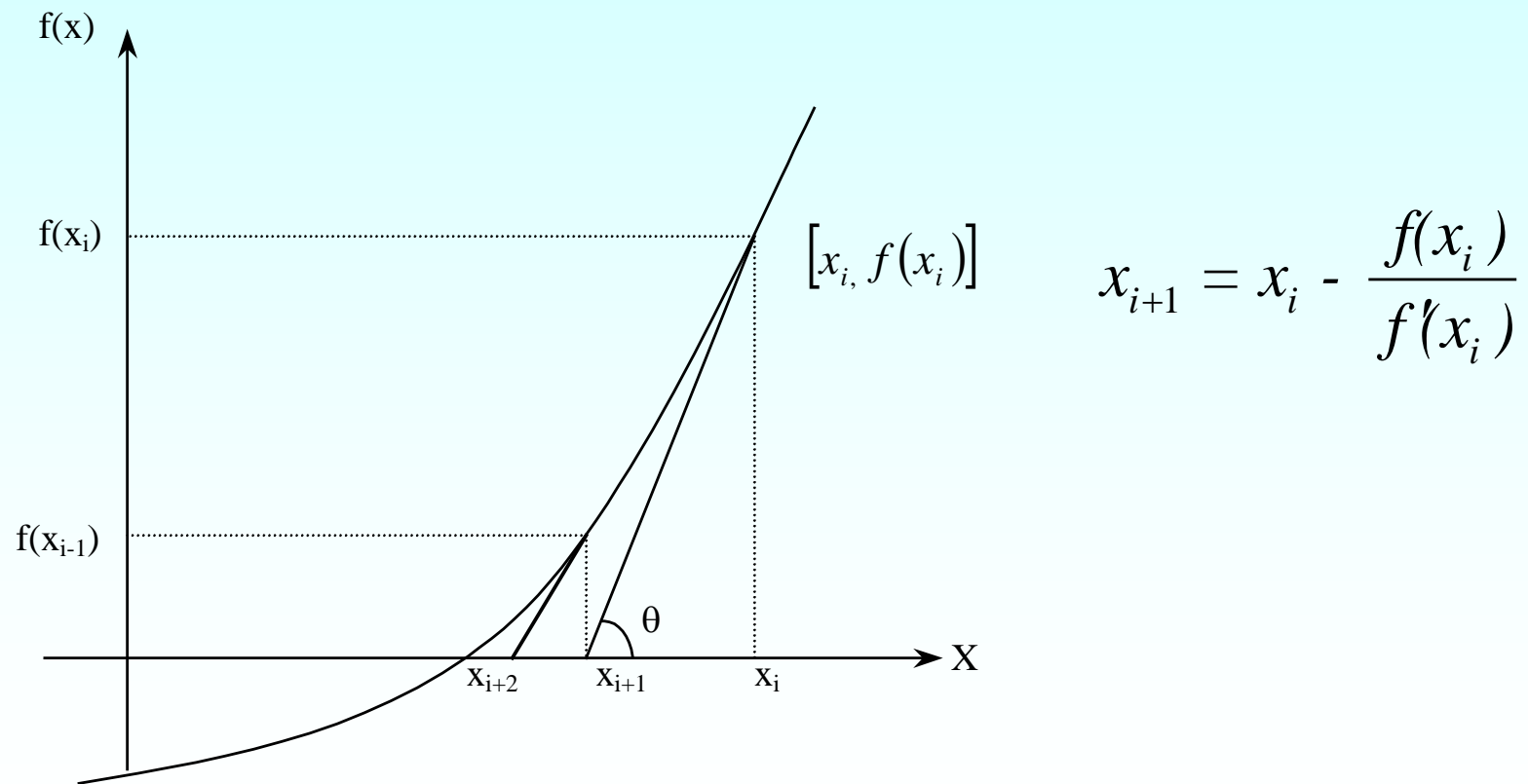
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Newton-Raphson Method

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Newton-Raphson Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$
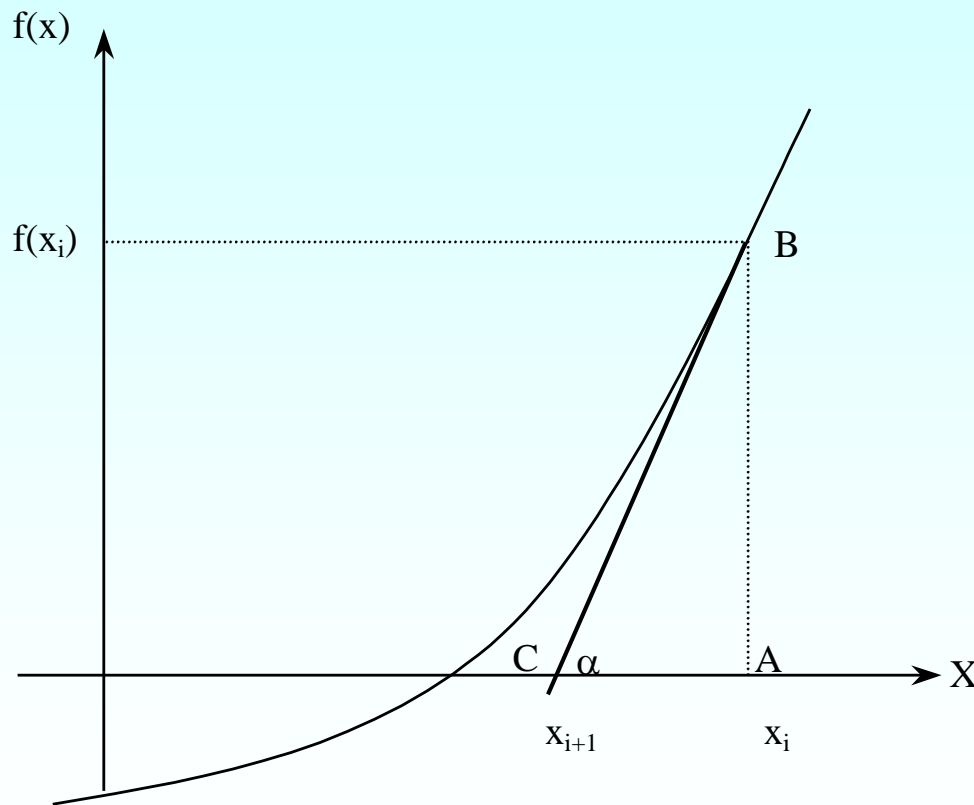
Figure 1 Geometrical illustration of the Newton-Raphson method.

Derivation

$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 2 Derivation of the Newton-Raphson method.

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Algorithm for Newton-Raphson Method

Step 1

Evaluate $f'(x)$ symbolically.

Step 2

Use an initial guess of the root, x_i , to estimate the new value of the root, x_{i+1} , as

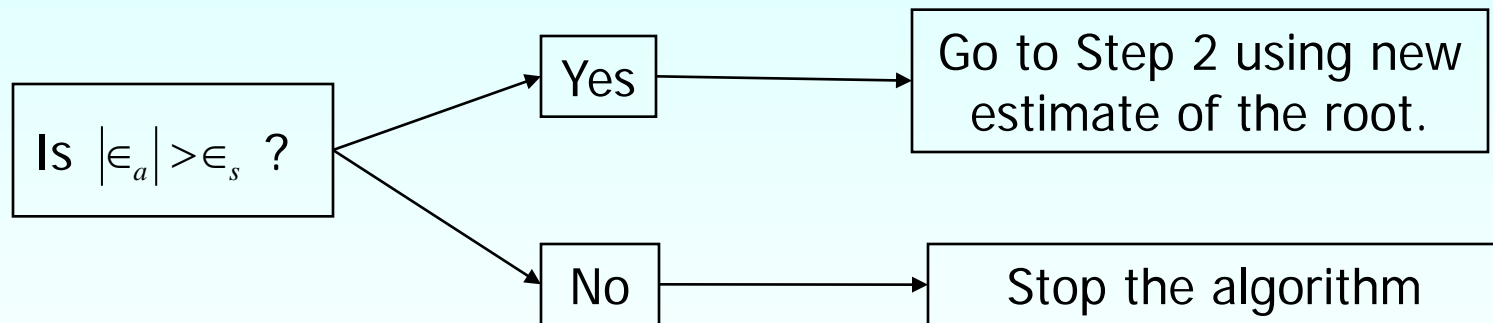$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Step 3

Find the absolute relative approximate error $\left| \in_a \right|$ as

$$\left| \in_a \right| = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100$$

Step 4

Compare the absolute relative approximate error with the pre-specified relative error tolerance $\in_s$.

Is $\left|\in_a\right| > \in_s$?

Yes → Go to Step 2 using new estimate of the root.

No → Stop the algorithm

Also, check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user.

Example 1

You are making a bookshelf to carry books that range from 8 ½ ″ to 11″ in height and would take 29″of space along length. The material is wood having Young's Modulus 3.667 Msi, thickness 3/8 ″ and width 12″. Your want to find the maximum vertical deflection of the bookshelf. The vertical deflection of the shelf is given by

$$v(x) = 0.42493 \text{ x } 10^{-4} x^3 - 0.13533 \text{ x } 10^{-8} x^5 - 0.66722 \text{ x } 10^{-6} x^4 - 0.018507x$$

where x is the position where the deflection is maximum. Hence to find the maximum deflection we need to find where $f(x) = \dfrac{dv}{dx} = 0$ and conduct the second derivative test.

Example 1 Cont.

The equation that gives the position 'x' where the deflection is maximum is given by

$$f(x) = -0.67665 \text{ x } 10^{-8} x^4 - 0.26689 \text{ x } 10^{-5} x^3 + 0.12748 \text{ x } 10^{-3} x^2 - 0.018507 = 0$$
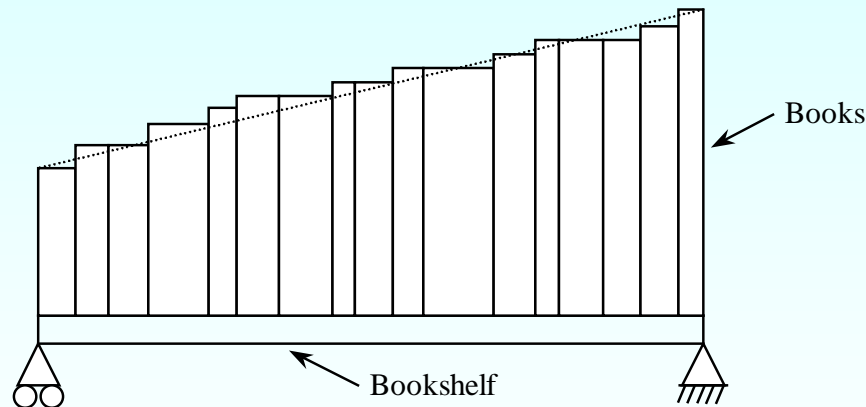
Figure 2 A loaded bookshelf.

Use the Newton-Raphson method of finding roots of equations to find the position where the deflection is maximum. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration, and the number of significant digits at least correct at the end of each iteration.

Example 1 Cont.
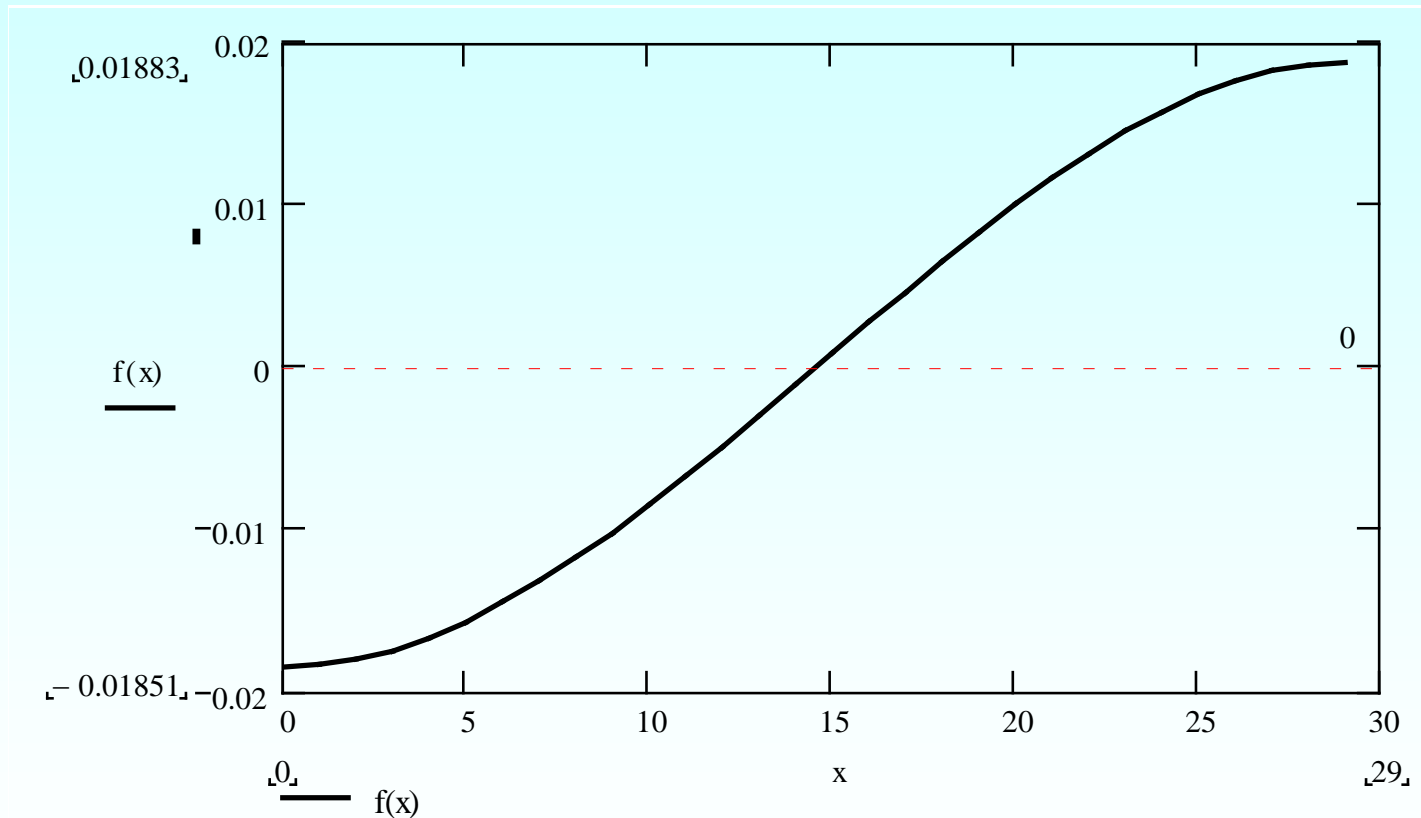
Figure 3 Graph of the function f(x).

$$f(x) = -0.67665 \times 10^{-8} x^4 - 0.26689 \times 10^{-5} x^3 + 0.12748 \times 10^{-3} x^2 - 0.018507 = 0$$

Example 1 Cont.

Solution

$$f(x) = -0.67665 \times 10^{-8} x^4 - 0.26689 \times 10^{-5} x^3 + 0.12748 \times 10^{-3} x^2 - 0.018507 = 0$$

$$f'(x) = -2.7066 \times 10^{-8} x^3 - 0.80067 \times 10^{-5} x^2 + 0.25496 \times 10^{-3} x = 0$$

Let us take the initial guess of the root of $f(x) = 0$ as $x_0 = 10$.

Iteration 1

The estimate of the root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 10 - \frac{-0.67665 \times 10^{-8} (10)^4 - 0.26689 \times 10^{-5} (10)^3 + 0.12748 \times 10^{-3} (10)^2 - 0.018507}{-2.7066 \times 10^{-8} (10)^3 - 0.80067 \times 10^{-5} (10)^2 + 0.25496 \times 10^{-3} (10)}$$

$$= 10 - \frac{-8.4956 \times 10^{-3}}{1.7219 \times 10^{-3}}$$

Example 1 Cont.

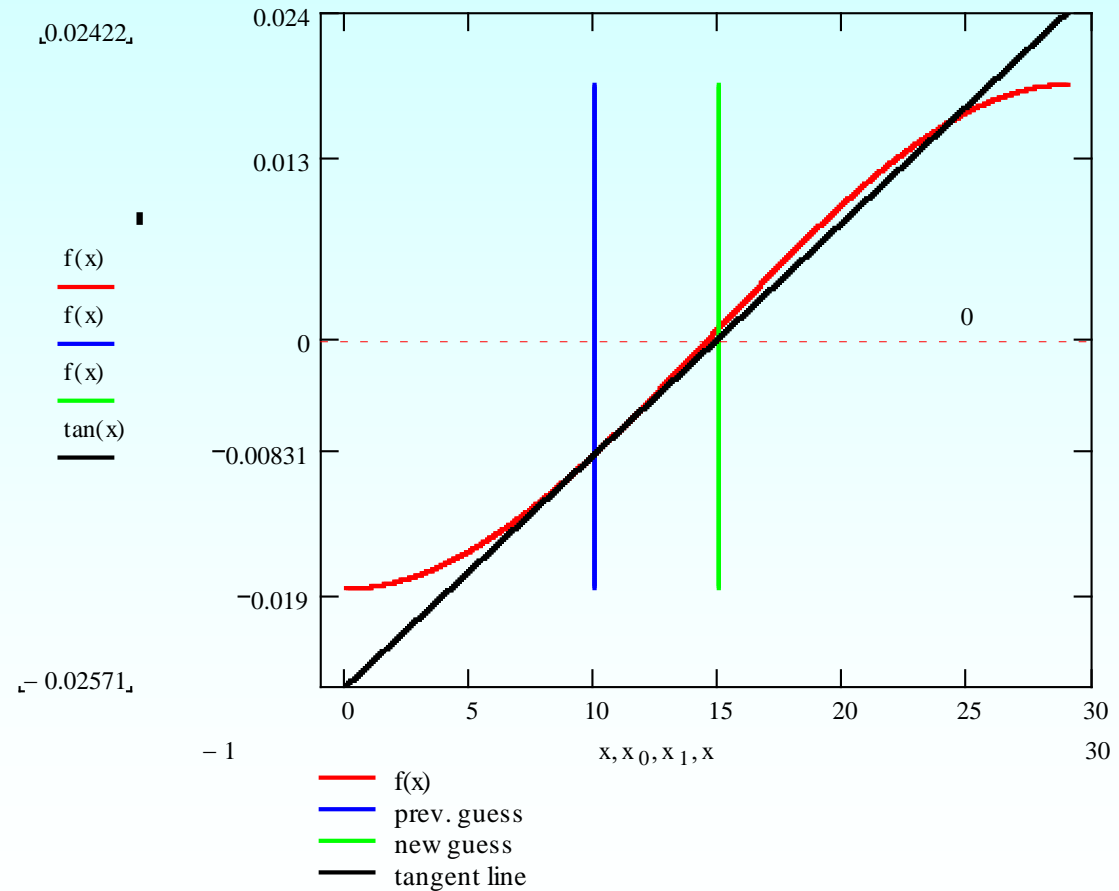$$= 10 - \left(- 4.9339 \right)$$
$$= 14.934$$

Figure 4 Graph of the estimate of the root after Iteration 1.

Example 1 Cont.

The absolute relative approximate error $|\epsilon_a|$ at the end of Iteration 1 is

$$|\epsilon_a| = \left| \frac{x_1 - x_0}{x_1} \right| \times 100$$

$$= \left| \frac{14.934 - 10}{14.934} \right| \times 100$$

$$= 33.038\%$$

The number of significant digits at least correct is 0, as you need an absolute relative approximate error of less than 5% for one significant digit to be correct in your result.
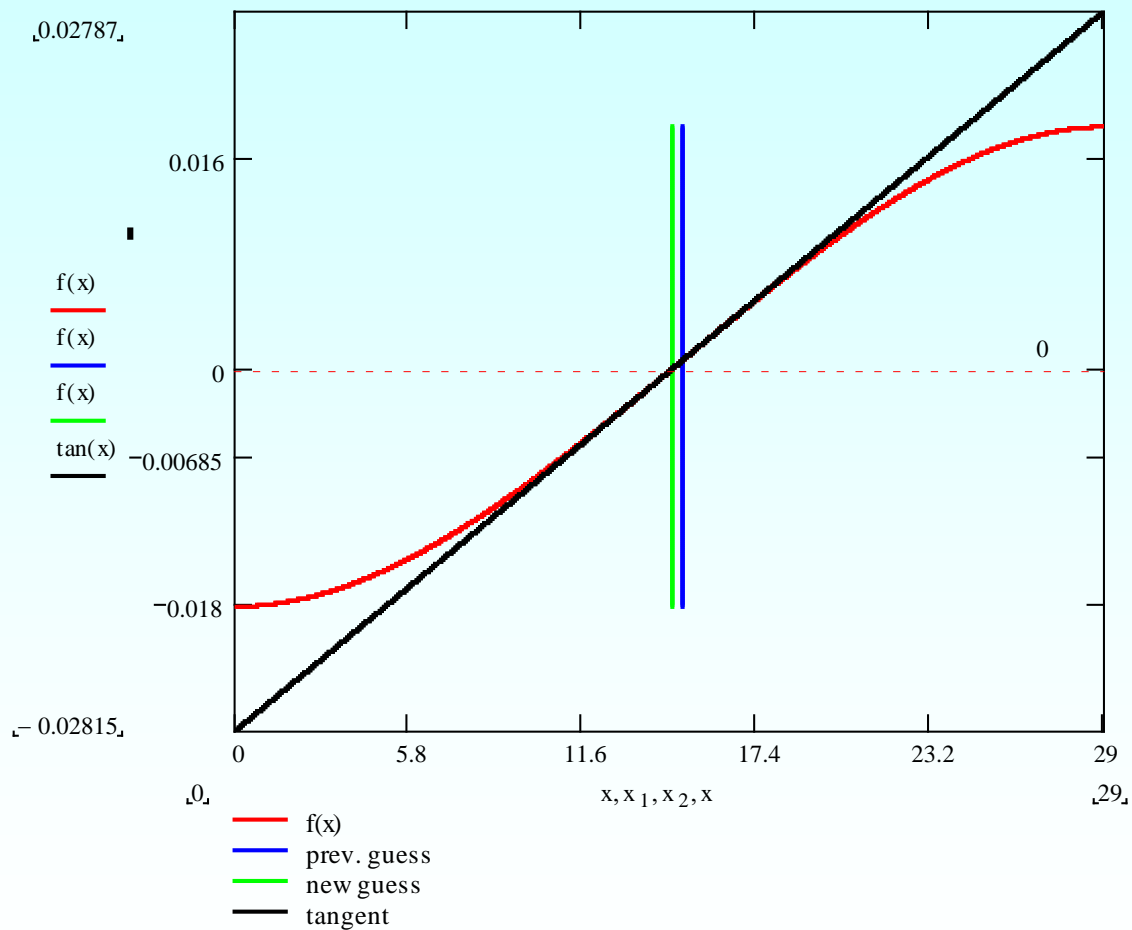
Example 1 Cont.

Iteration 2

The estimate of the root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 14.934 - \frac{\left(\begin{array}{l} -0.67665\times10^{-8}(14.934)^4 - 0.26689\times10^{-5}(14.934)^3 \\ +0.12748\times10^{-3}(14.934)^2 - 0.018507 \end{array}\right)}{\left(\begin{array}{l} -2.7066\times10^{-8}(14.934)^3 - 0.80067\times10^{-5}(14.934)^2 \\ +0.25496\times10^{-3}(14.934) \end{array}\right)}$$

$$= 14.934 - \frac{6.9829\times10^{-4}}{1.9317\times10^{-3}}$$

$$= 14.934 - (0.36149)$$

$$= 14.572$$

Example 1 Cont.

Figure 5 Graph of the estimate of the root after Iteration 2.

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Example 1 Cont.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 2 is

$$\left|\in_a\right| = \left|\frac{x_2 - x_1}{x_2}\right| \times 100$$

$$= \left|\frac{14.572 - 14.934}{14.572}\right| \times 100$$

$$= 2.4806\ \%$$

The number of significant digits at least correct is 1, because the absolute relative approximate error is less than 5%.

Example 1 Cont.

Iteration 3

The estimate of the root is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 14.572 - \frac{\left(\begin{array}{l} -0.67665 \times 10^{-8}(14.572)^4 - 0.26689 \times 10^{-5}(14.572)^3 \\ +0.12748 \times 10^{-3}(14.572)^2 - 0.018507 \end{array}\right)}{\left(\begin{array}{l} -2.7066 \times 10^{-8}(14.572)^3 - 0.80067 \times 10^{-5}(14.572)^2 \\ +0.25496 \times 10^{-3}(14.572) \end{array}\right)}$$

$$= 14.572 - \frac{-4.7078 \times 10^{-9}}{1.9314 \times 10^{-3}}$$

$$= 14.572 - \left(-2.4375 \times 10^{-6}\right)$$

$$= 14.572$$
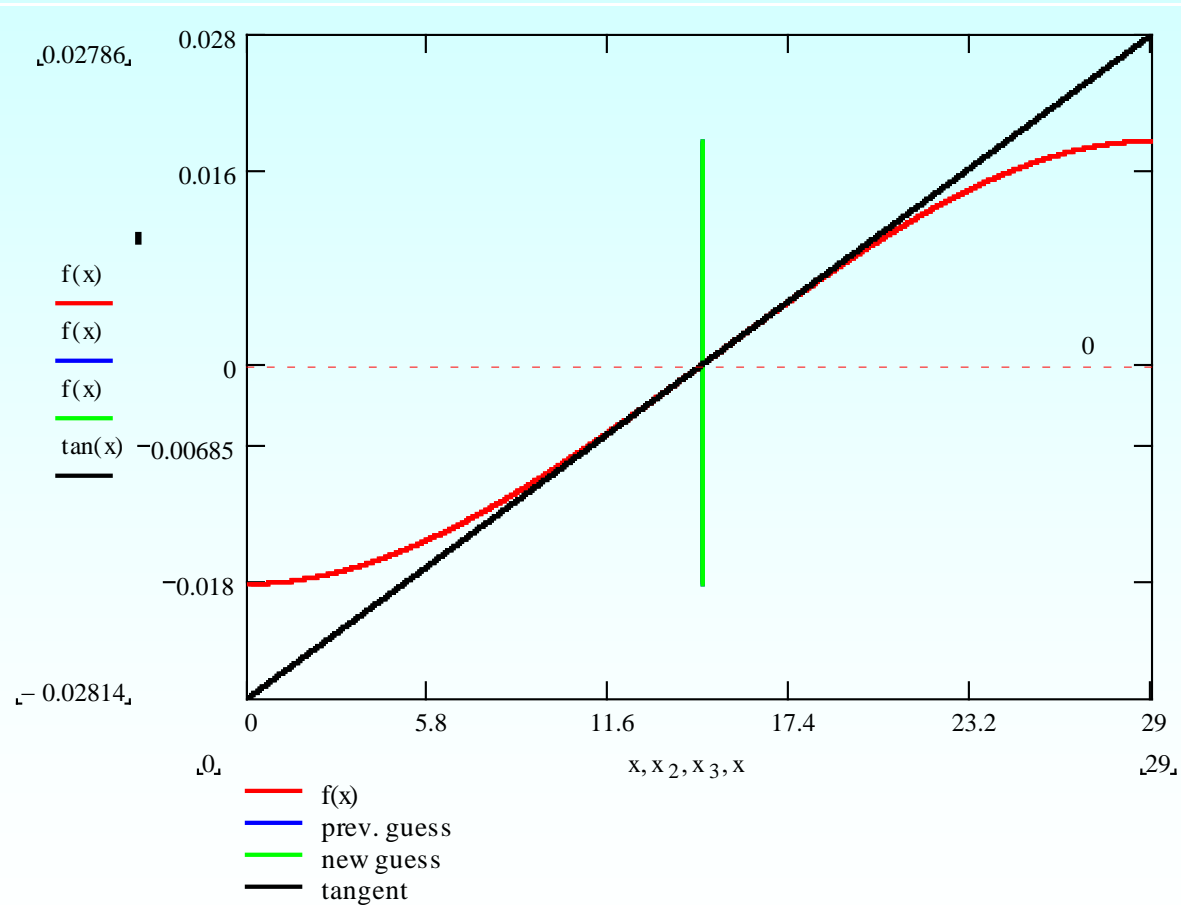
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Example 1 Cont.

Figure 6 Graph of the estimate of the root after Iteration 3.

Example 1 Cont.

The absolute relative approximate error $\left| \in_a \right|$ at the end of Iteration 3 is

$$\left| \in_a \right| = \left| \frac{x_2 - x_1}{x_2} \right| \times 100$$

$$= \left| \frac{14.572 - 14.572}{14.572} \right| \times 100$$

$$= 1.6727 \times 10^{-5} \, \%$$

Example 1 Cont.

Hence the number if significant digits at least correct is given by the largest value of m for which

$$\left| \in_a \right| \leq 0.5 \times 10^{2-m}$$

$$1.6727 \times 10^{-5} \leq 0.5 \times 10^{2-m}$$

$$3.3454 \times 10^{-5} \leq 10^{2-m}$$

$$\log\left(3.3454 \times 10^{-5}\right) \leq 2 - m$$

$$m \leq 2 - \log\left(3.3454 \times 10^{-5}\right) = 6.4756$$

So

$$m = 6$$

The number of significant digits at least correct in the estimated root 14.572 is 6.

Advantages and Drawbacks of Newton Raphson Method

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Advantages

- Converges fast (quadratic convergence), if it converges.
- Requires only one guess

Drawbacks

1. Divergence at inflection points
   Selection of the initial guess or an iteration value of the root that is close to the inflection point of the function $f(x)$ may start diverging away from the root in ther Newton-Raphson method.

For example, to find the root of the equation $f(x) = (x-1)^3 + 0.512 = 0$.

The Newton-Raphson method reduces to $x_{i+1} = x_i - \dfrac{(x_i^3 - 1)^3 + 0.512}{3(x_i - 1)^2}$.

Table 1 shows the iterated values of the root of the equation.

The root starts to diverge at Iteration 6 because the previous estimate of 0.92589 is close to the inflection point of $x = 1$.

Eventually after 12 more iterations the root converges to the exact value of $x = 0.2$.

Drawbacks – Inflection Points

Table 1 Divergence near inflection point.

Iteration Number	x_i
0	5.0000
1	3.6560
2	2.7465
3	2.1084
4	1.6000
5	0.92589
6	−30.119
7	−19.746
18	0.2000
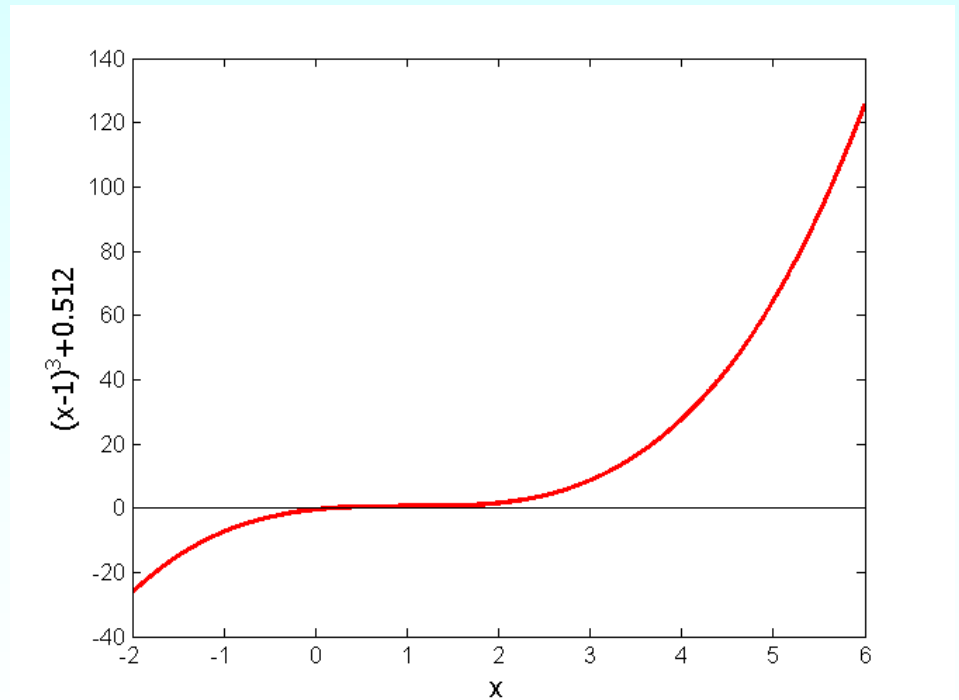
Figure 8 Divergence at inflection point for

$$f(x) = (x-1)^3 + 0.512 = 0$$

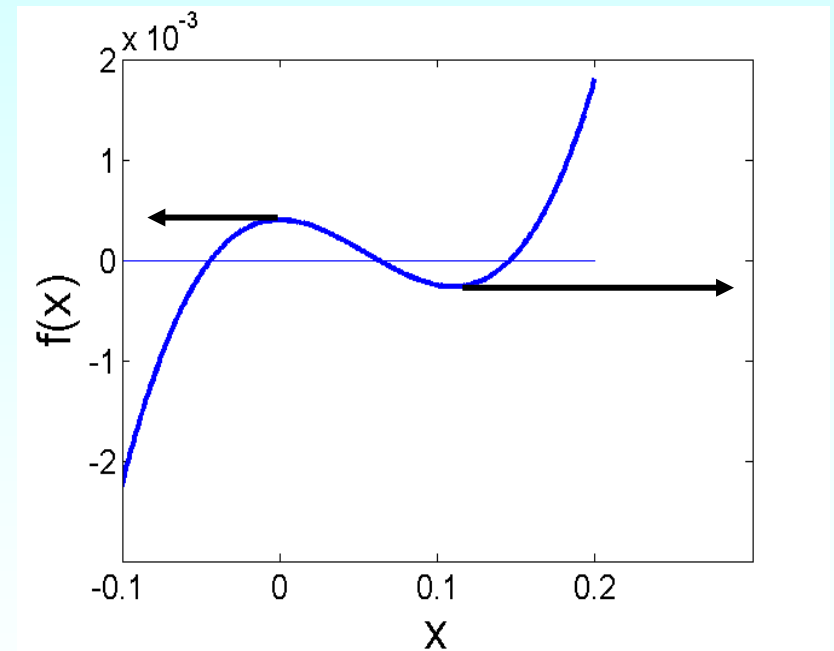# Drawbacks – Division by Zero

2. Division by zero
   For the equation

$$f(x) = x^3 - 0.03x^2 + 2.4 \times 10^{-6} = 0$$

the Newton-Raphson method reduces to

$$x_{i+1} = x_i - \frac{x_i^3 - 0.03x_i^2 + 2.4 \times 10^{-6}}{3x_i^2 - 0.06x_i}$$

For $x_0 = 0$ or $x_0 = 0.02$, the denominator will equal zero.

Figure 9 Pitfall of division by zero or near a zero number

Drawbacks – Oscillations near local maximum and minimum

3. <u>Oscillations near local maximum and minimum</u>

Results obtained from the Newton-Raphson method may oscillate about the local maximum or minimum without converging on a root but converging on the local maximum or minimum.

Eventually, it may lead to division by a number close to zero and may diverge.

For example for $f(x) = x^2 + 2 = 0$ the equation has no real roots.

Drawbacks – Oscillations near local maximum and minimum

Table 3 Oscillations near local maxima and mimima in Newton-Raphson method.

Iteration Number	x_i	$f(x_i)$	$\left|\in_a\right|\%$
0	−1.0000	3.00	
1	0.5	2.25	300.00
2	−1.75	5.063	128.571
3	−0.30357	2.092	476.47
4	3.1423	11.874	109.66
5	1.2529	3.570	150.80
6	−0.17166	2.029	829.88
7	5.7395	34.942	102.99
8	2.6955	9.266	112.93
9	0.97678	2.954	175.96
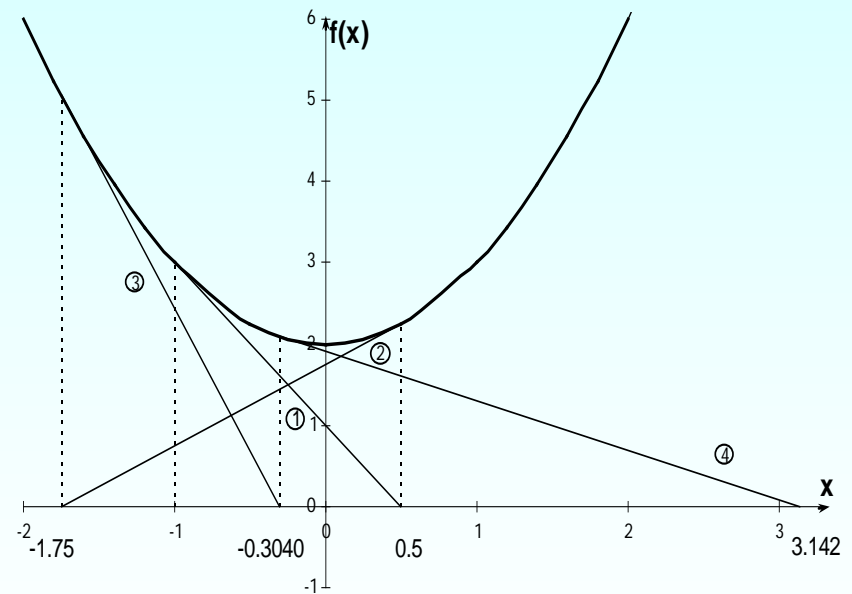
Figure 10 Oscillations around local minima for $f(x) = x^2 + 2$.

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Drawbacks – Root Jumping

4. <u>Root Jumping</u>
In some cases where the function $f(x)$ is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root.
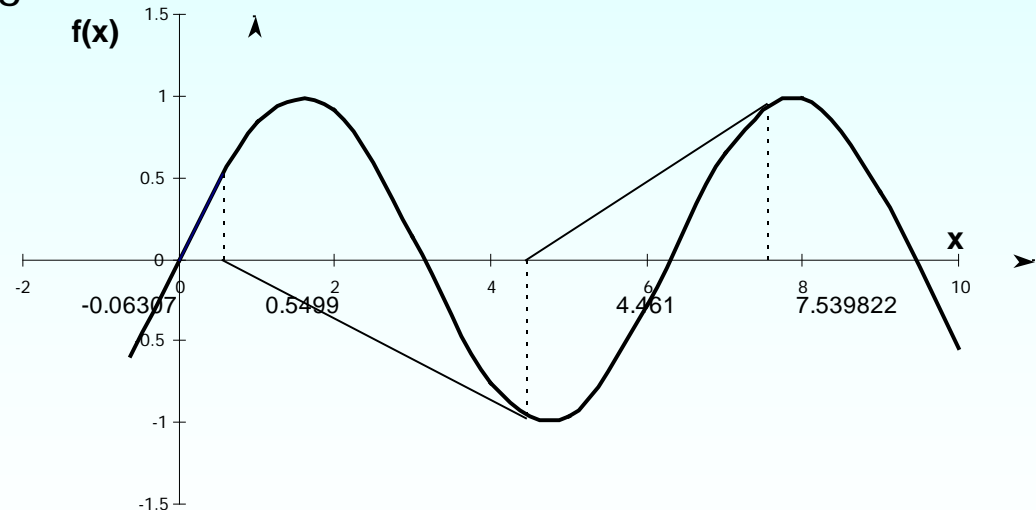
For example
$$f(x) = \sin x = 0$$

Choose
$$x_0 = 2.4\pi = 7.539822$$

It will converge to $x = 0$

instead of $x = 2\pi = 6.2831853$

Figure 11 Root jumping from intended location of root for $f(x) = \sin x = 0$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/newton_raphson.html

THE END