

## Chapter 05.04

# Lagrangian Interpolation

After reading this chapter, you should be able to:

1. derive Lagrangian method of interpolation,
2. solve problems using Lagrangian method of interpolation, and
3. use Lagrangian interpolants to find derivatives and integrals of discrete functions.

### What is interpolation?

Many times, data is given only at discrete points such as  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $\dots$ ,  $(x_{n-1}, y_{n-1})$ ,  $(x_n, y_n)$ . So, how then does one find the value of  $y$  at any other value of  $x$ ? Well, a continuous function  $f(x)$  may be used to represent the  $n + 1$  data values with  $f(x)$  passing through the  $n + 1$  points (Figure 1). Then one can find the value of  $y$  at any other value of  $x$ . This is called *interpolation*.

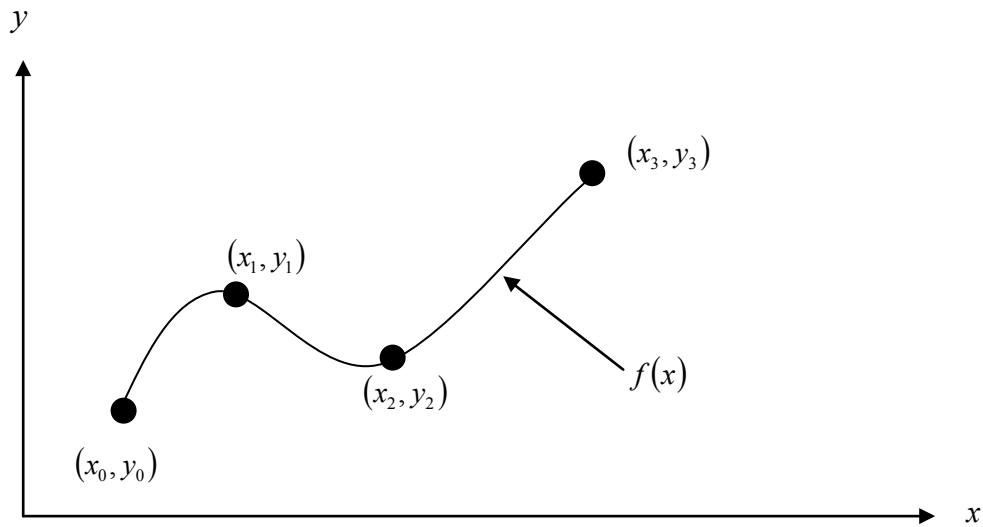
Of course, if  $x$  falls outside the range of  $x$  for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function  $f(x)$  should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

- (A) evaluate,
- (B) differentiate, and
- (C) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order  $n$  that passes through the  $n + 1$  data points. One of the methods used to find this polynomial is called the Lagrangian method of interpolation. Other methods include Newton's divided difference polynomial method and the direct method. We discuss the Lagrangian method in this chapter.



**Figure 1** Interpolation of discrete data.

The Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i)$$

where  $n$  in  $f_n(x)$  stands for the  $n^{\text{th}}$  order polynomial that approximates the function  $y = f(x)$  given at  $n + 1$  data points as  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ , and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

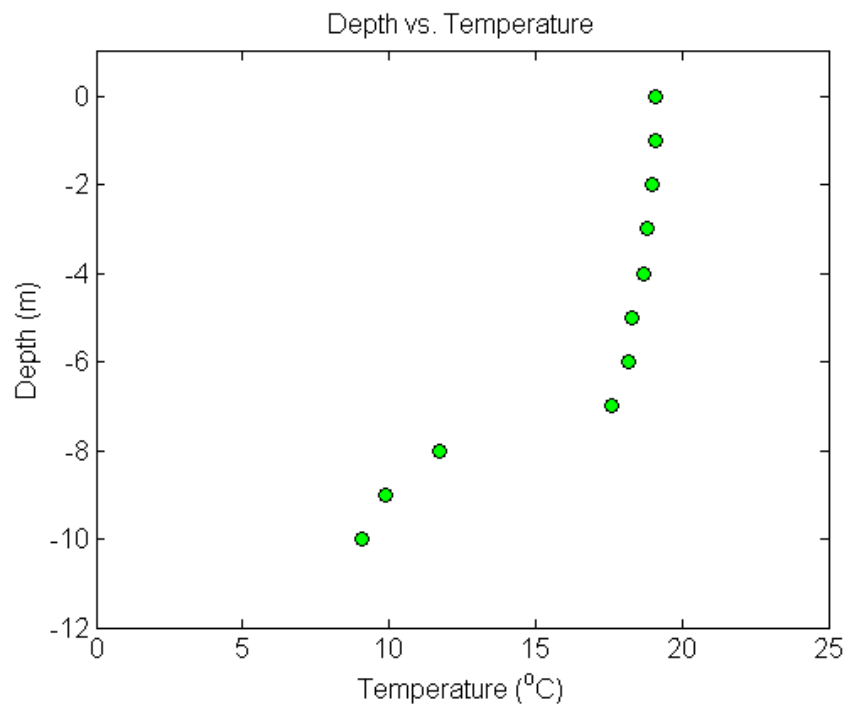
$L_i(x)$  is a weighting function that includes a product of  $n - 1$  terms with terms of  $j = i$  omitted. The application of Lagrangian interpolation will be clarified using an example.

### Example 1

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth data for a lake in Table 1.

**Table 1** Temperature vs. depth for a lake.

Temperature, $T$ ( $^{\circ}\text{C}$ )	Depth, $z$ (m)
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10

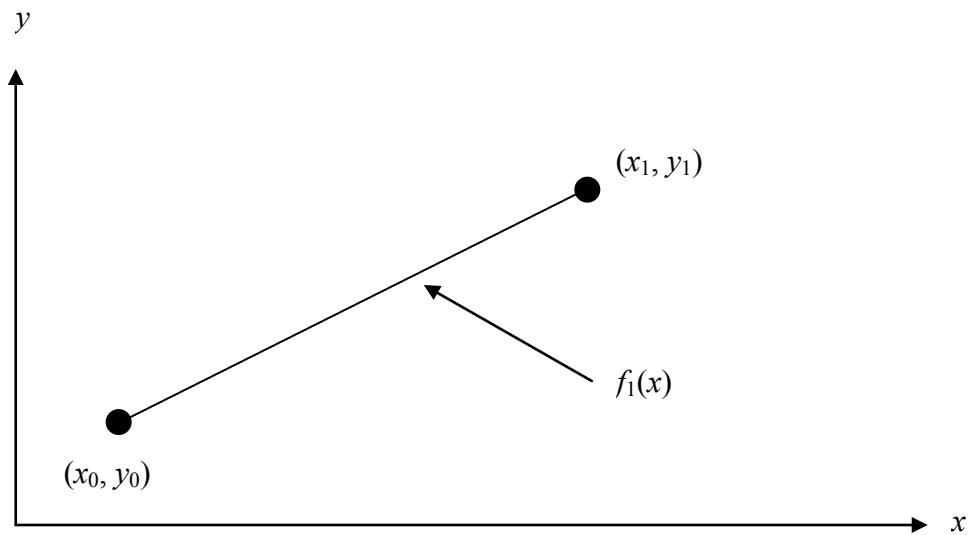
**Figure 2** Temperature vs. depth of a lake.

Using the given data, we see the largest change in temperature is between  $z = -8$  m and  $z = -7$  m. Determine the value of the temperature at  $z = -7.5$  m using a first order Lagrange polynomial.

**Solution**

For first order Lagrange polynomial interpolation (also called linear interpolation), the temperature is given by

$$\begin{aligned} T(z) &= \sum_{i=0}^1 L_i(z)T(z_i) \\ &= L_0(z)T(z_0) + L_1(z)T(z_1) \end{aligned}$$



**Figure 3** Linear interpolation.

Since we want to find the temperature at  $z = -7.5$  m, we need to choose the two data points that are closest to  $z = -7.5$  m that also bracket  $z = -7.5$  m to evaluate it. The two points are  $z_0 = -8$  and  $z_1 = -7$ .

Then

$$z_0 = -8, T(z_0) = 11.7$$

$$z_1 = -7, T(z_1) = 17.6$$

gives

$$\begin{aligned} L_0(z) &= \prod_{\substack{j=0 \\ j \neq 0}}^1 \frac{z - z_j}{z_0 - z_j} \\ &= \frac{z - z_1}{z_0 - z_1} \end{aligned}$$

$$L_1(z) = \prod_{\substack{j=0 \\ j \neq 1}}^1 \frac{z - z_j}{z_1 - z_j}$$

$$= \frac{z - z_0}{z_1 - z_0}$$

Hence

$$\begin{aligned} T(z) &= \frac{z - z_1}{z_0 - z_1} T(z_0) + \frac{z - z_0}{z_1 - z_0} T(z_1) \\ &= \frac{z + 7}{-8 + 7} (11.7) + \frac{z + 8}{-7 + 8} (17.6), \quad -8 \leq z \leq -7 \\ T(-7.5) &= \frac{-7.5 + 7}{-8 + 7} (11.7) + \frac{-7.5 + 8}{-7 + 8} (17.6) \\ &= 0.5(11.7) + 0.5(17.6) \\ &= 14.65^\circ\text{C} \end{aligned}$$

You can see that  $L_0(z) = 0.5$  and  $L_1(z) = 0.5$  are like weightages given to the temperatures at  $z = -8$  m and  $z = -7$  m to calculate the temperature at  $z = -7.5$  m.

### Quadratic Interpolation

#### **Example 2**

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth data for a lake in Table 2.

**Table 2** Temperature vs. depth for a lake.

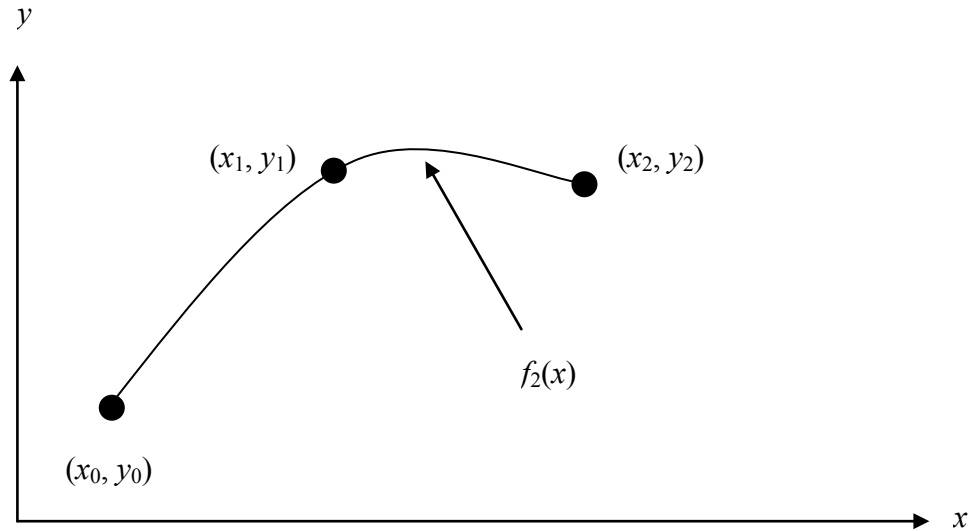
Temperature, $T$ ( $^\circ\text{C}$ )	Depth, $z$ (m)
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10

Using the given data, we see the largest change in temperature is between  $z = -8$  m and  $z = -7$  m. Determine the value of the temperature at  $z = -7.5$  m using a second order Lagrange polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

**Solution**

For second order Lagrange polynomial interpolation (also called quadratic interpolation), the temperature is given by

$$\begin{aligned} T(z) &= \sum_{i=0}^2 L_i(z)T(z_i) \\ &= L_0(z)T(z_0) + L_1(z)T(z_1) + L_2(z)T(z_2) \end{aligned}$$



**Figure 4** Quadratic interpolation.

Since we want to find the temperature at  $z = -7.5$ , we need to choose the three data points that are closest to  $z = -7.5$  that also bracket  $z = -7.5$  to evaluate it. These three points are  $z_0 = -9$ ,  $z_1 = -8$  and  $z_2 = -7$ . (Choosing the three points as  $z_0 = -8$ ,  $z_1 = -7$  and  $z_2 = -6$  is equally valid.)

Then

$$z_0 = -9, \quad T(z_0) = 9.9$$

$$z_1 = -8, \quad T(z_1) = 11.7$$

$$z_2 = -7, \quad T(z_2) = 17.6$$

gives

$$\begin{aligned} L_0(z) &= \prod_{\substack{j=0 \\ j \neq 0}}^2 \frac{z - z_j}{z_0 - z_j} \\ &= \left( \frac{z - z_1}{z_0 - z_1} \right) \left( \frac{z - z_2}{z_0 - z_2} \right) \end{aligned}$$

$$L_1(z) = \prod_{\substack{j=0 \\ j \neq 1}}^2 \frac{z - z_j}{z_1 - z_j}$$

$$\begin{aligned}
 &= \left( \frac{z - z_0}{z_1 - z_0} \right) \left( \frac{z - z_2}{z_1 - z_2} \right) \\
 L_2(z) &= \prod_{\substack{j=0 \\ j \neq 2}}^2 \frac{z - z_j}{z_2 - z_j} \\
 &= \left( \frac{z - z_0}{z_2 - z_0} \right) \left( \frac{z - z_1}{z_2 - z_1} \right)
 \end{aligned}$$

Hence

$$T(z) = \left( \frac{z - z_1}{z_0 - z_1} \right) \left( \frac{z - z_2}{z_0 - z_2} \right) T(z_0) + \left( \frac{z - z_0}{z_1 - z_0} \right) \left( \frac{z - z_2}{z_1 - z_2} \right) T(z_1) + \left( \frac{z - z_0}{z_2 - z_0} \right) \left( \frac{z - z_1}{z_2 - z_1} \right) T(z_2),$$

$z_0 \leq z \leq z_2$

$$\begin{aligned}
 T(-7.5) &= \frac{(-7.5 + 8)(-7.5 + 7)}{(-9 + 8)(-9 + 7)} (9.9) + \frac{(-7.5 + 9)(-7.5 + 7)}{(-8 + 9)(-8 + 7)} (11.7) \\
 &\quad + \frac{(-7.5 + 9)(-7.5 + 8)}{(-7 + 9)(-7 + 8)} (17.6) \\
 &= (-0.125)(9.9) + (0.75)(11.7) + (0.375)(17.6) \\
 &= 14.138^\circ\text{C}
 \end{aligned}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the first and second order polynomial is

$$\begin{aligned}
 |\epsilon_a| &= \left| \frac{14.138 - 14.65}{14.138} \right| \times 100 \\
 &= 3.6251\%
 \end{aligned}$$

### Example 3

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth data for a lake in Table 3.

**Table 3** Temperature vs. depth for a lake.

Temperature, $T$ ( $^{\circ}\text{C}$ )	Depth, $z$ (m)
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10

Using the given data, we see the largest change in temperature is between  $z = -8$  m and  $z = -7$  m.

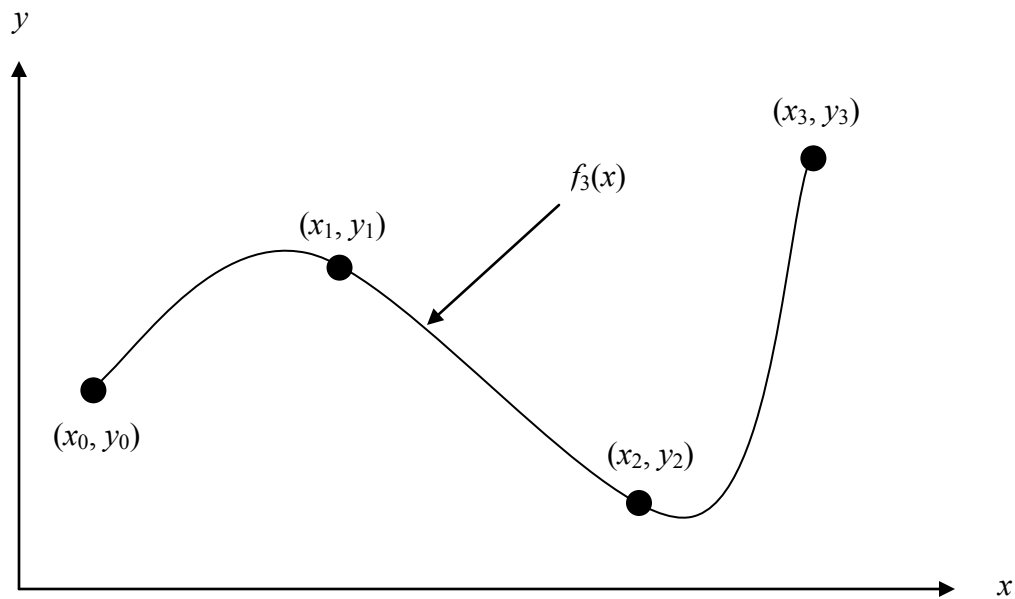
- Determine the value of the temperature at  $z = -7.5$  m using a third order Lagrange polynomial. Find the absolute relative approximate error for the third order polynomial approximation.
- The position where the thermocline exists is given where  $\frac{d^2T}{dz^2} = 0$ . Using the expression from part (a), what is the value of the depth at which the thermocline exists?

### Solution

a) For third order Lagrange polynomial interpolation (also called cubic interpolation), we choose the temperature given by

$$\begin{aligned}
 T(z) &= \sum_{i=0}^3 L_i(z)T(z_i) \\
 &= L_0(z)T(z_0) + L_1(z)T(z_1) + L_2(z)T(z_2) + L_3(z)T(z_3)
 \end{aligned}$$





**Figure 5** Cubic interpolation.

Since we want to find the temperature at  $z = -7.5$ , we need to choose the four data points that are closest to  $z = -7.5$  that also bracket  $z = -7.5$  to evaluate it. The four data points are  $z_0 = -9$ ,  $z_1 = -8$ ,  $z_2 = -7$  and  $z_3 = -6$ .

Then

$$z_0 = -9, T(z_0) = 9.9$$

$$z_1 = -8, T(z_1) = 11.7$$

$$z_2 = -7, T(z_2) = 17.6$$

$$z_3 = -6, T(z_3) = 18.2$$

gives

$$\begin{aligned} L_0(z) &= \prod_{\substack{j=0 \\ j \neq 0}}^3 \frac{z - z_j}{z_0 - z_j} \\ &= \left( \frac{z - z_1}{z_0 - z_1} \right) \left( \frac{z - z_2}{z_0 - z_2} \right) \left( \frac{z - z_3}{z_0 - z_3} \right) \end{aligned}$$

$$\begin{aligned} L_1(z) &= \prod_{\substack{j=0 \\ j \neq 1}}^3 \frac{z - z_j}{z_1 - z_j} \\ &= \left( \frac{z - z_0}{z_1 - z_0} \right) \left( \frac{z - z_2}{z_1 - z_2} \right) \left( \frac{z - z_3}{z_1 - z_3} \right) \end{aligned}$$

$$L_2(z) = \prod_{\substack{j=0 \\ j \neq 2}}^3 \frac{z - z_j}{z_2 - z_j}$$

$$= \left( \frac{z - z_0}{z_2 - z_0} \right) \left( \frac{z - z_1}{z_2 - z_1} \right) \left( \frac{z - z_3}{z_2 - z_3} \right)$$

$$L_3(z) = \prod_{\substack{j=0 \\ j \neq 3}}^3 \frac{z - z_j}{z_3 - z_j}$$

$$= \left( \frac{z - z_0}{z_3 - z_0} \right) \left( \frac{z - z_1}{z_3 - z_1} \right) \left( \frac{z - z_2}{z_3 - z_2} \right)$$

Hence

$$T(z) = \left( \frac{z - z_1}{z_0 - z_1} \right) \left( \frac{z - z_2}{z_0 - z_2} \right) \left( \frac{z - z_3}{z_0 - z_3} \right) T(z_0) + \left( \frac{z - z_0}{z_1 - z_0} \right) \left( \frac{z - z_2}{z_1 - z_2} \right) \left( \frac{z - z_3}{z_1 - z_3} \right) T(z_1)$$

$$+ \left( \frac{z - z_0}{z_2 - z_0} \right) \left( \frac{z - z_1}{z_2 - z_1} \right) \left( \frac{z - z_3}{z_2 - z_3} \right) T(z_2) + \left( \frac{z - z_0}{z_3 - z_0} \right) \left( \frac{z - z_1}{z_3 - z_1} \right) \left( \frac{z - z_2}{z_3 - z_2} \right) T(z_3)$$

$z_0 \leq z \leq z_3$

$$T(-7.5) = \frac{(-7.5 + 8)(-7.5 + 7)(-7.5 + 6)}{(-9 + 8)(-9 + 7)(-9 + 6)} (9.9) + \frac{(-7.5 + 9)(-7.5 + 7)(-7.5 + 6)}{(-8 + 9)(-8 + 7)(-8 + 6)} (11.7)$$

$$+ \frac{(-7.5 + 9)(-7.5 + 8)(-7.5 + 6)}{(-7 + 9)(-7 + 8)(-7 + 6)} (17.6) + \frac{(-7.5 + 9)(-7.5 + 8)(-7.5 + 7)}{(-6 + 9)(-6 + 8)(-6 + 7)} (18.2)$$

$$= (-0.0625)(9.9) + (0.5625)(11.7) + (0.5625)(17.6) + (-0.0625)(18.2)$$

$$= 14.725^\circ\text{C}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the second and third order polynomial is

$$|\epsilon_a| = \left| \frac{14.725 - 14.138}{14.725} \right| \times 100$$

$$= 3.9898\%$$

b) To find the position of the thermocline, we must find the points of inflection of the third order polynomial, given by  $\frac{d^2T}{dz^2} = 0$ .

$$T(z) = \frac{(z + 8)(z + 7)(z + 6)}{(-9 + 8)(-9 + 7)(-9 + 6)} (9.9) + \frac{(z + 9)(z + 7)(z + 6)}{(-8 + 9)(-8 + 7)(-8 + 6)} (11.7)$$

$$+ \frac{(z + 9)(z + 8)(z + 6)}{(-7 + 9)(-7 + 8)(-7 + 6)} (17.6) + \frac{(z + 9)(z + 8)(z + 7)}{(-6 + 9)(-6 + 8)(-6 + 7)} (18.2)$$

$-9 \leq z \leq -6$

$$= -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \quad -9 \leq z \leq -6$$

$$\frac{dT}{dz} = -262.58 - 71.1z - 4.7z^2, \quad -9 \leq z \leq -6$$

$$\frac{d^2T}{dz^2} = -71.1 - 9.4z, \quad -9 \leq z \leq -6$$

Simply setting this expression equal to zero, we get

$$0 = -71.1 - 9.4z, \quad -9 \leq z \leq -6$$

$$z = -7.5638 \text{ m}$$

This answer can be verified due to the fact that it falls within the specified range of the third order polynomial and it also falls within the region of the greatest temperature change in the collected data from the lake.

---

#### INTERPOLATION

---

Topic	Lagrange Interpolation
Summary	Textbook notes on the Lagrangian method of interpolation
Major	Civil Engineering
Authors	Autar Kaw, Michael Keteltas
Last Revised	November 8, 2012
Web Site	<a href="http://numericalmethods.eng.usf.edu">http://numericalmethods.eng.usf.edu</a>

---