Romberg Rule of Integration

Civil Engineering Majors

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Basis of Romberg Rule

Integration

The process of measuring the area under a curve.

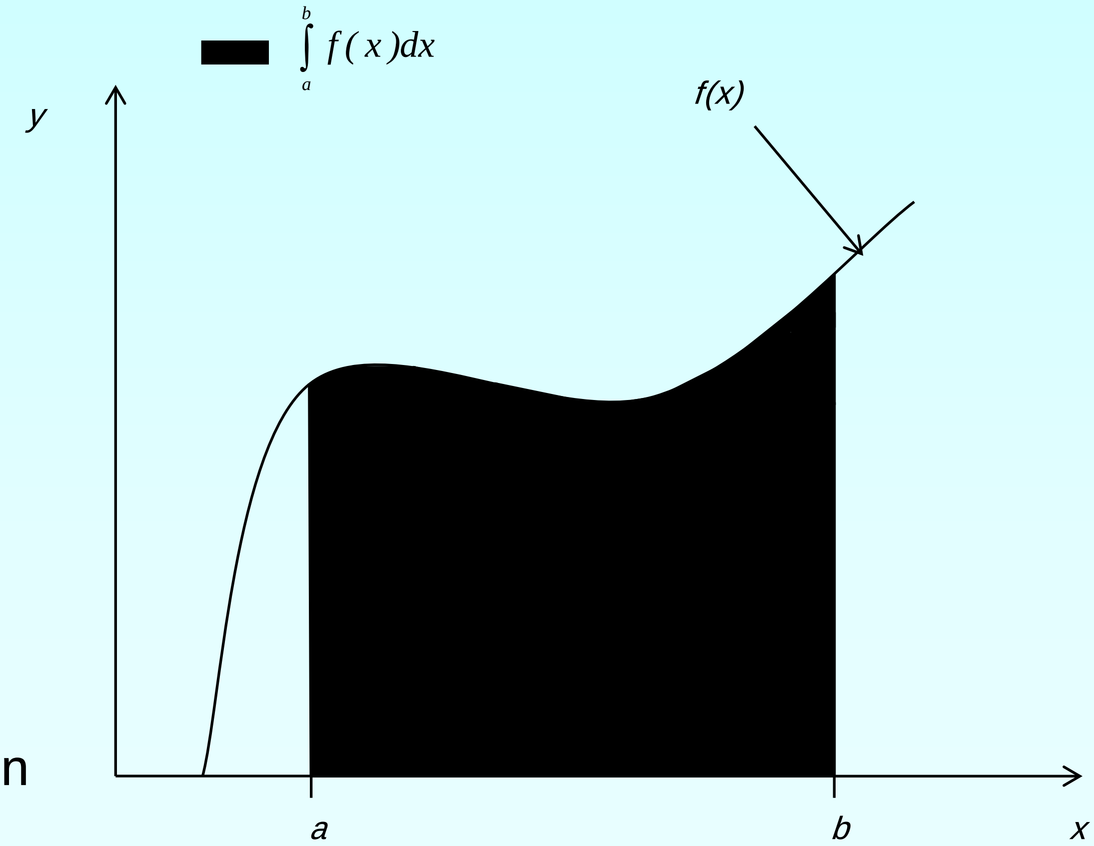
$$I = \int_a^b f(x)dx$$

Where:

f(x) is the integrand

a= lower limit of integration

b= upper limit of integration



What is The Romberg Rule?

Romberg Integration is an extrapolation formula of the Trapezoidal Rule for integration. It provides a better approximation of the integral by reducing the True Error.

Error in Multiple Segment Trapezoidal Rule

The true error in a multiple segment Trapezoidal Rule with n segments for an integral

$$I = \int_a^b f(x)dx$$

Is given by

$$E_t = \frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$$

where for each *i*, ξ_i is a point somewhere in the domain , $[a+(i-1)h, a+ih]$.

Error in Multiple Segment Trapezoidal Rule

The term $\dfrac{\sum_{i=1}^{n} f''(\xi_i)}{n}$ can be viewed as an approximate average value of $f''(x)$ in $[a,b]$.

This leads us to say that the true error, E_t previously defined can be approximated as

$$E_t \cong \alpha \frac{1}{n^2}$$

Error in Multiple Segment Trapezoidal Rule

Table 1 shows the results obtained for the integral using multiple segment Trapezoidal rule for

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

n	Value	E_t	$\lvert \in_t \rvert \%$	$\lvert \in_a \rvert \%$
1	11868	807	7.296	---
2	11266	205	1.854	5.343
3	11153	91.4	0.8265	1.019
4	11113	51.5	0.4655	0.3594
5	11094	33.0	0.2981	0.1669
6	11084	22.9	0.2070	0.09082
7	11078	16.8	0.1521	0.05482
8	11074	12.9	0.1165	0.03560

Table 1: Multiple Segment Trapezoidal Rule Values

Error in Multiple Segment Trapezoidal Rule

The true error gets approximately quartered as the number of segments is doubled. This information is used to get a better approximation of the integral, and is the basis of Richardson's extrapolation.

Richardson's Extrapolation for Trapezoidal Rule

The true error, E_t in the *n*-segment Trapezoidal rule is estimated as

$$E_t \approx \frac{C}{n^2}$$

where *C* is an *approximate constant* of proportionality. Since

$$E_t = TV - I_n$$

Where TV = true value and I_n = approx. value

Richardson's Extrapolation for Trapezoidal Rule

From the previous development, it can be shown that

$$\frac{C}{(2n)^2} \approx TV - I_{2n}$$

when the segment size is doubled and that

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

which is Richardson's Extrapolation.

Example 1

The concentration of benzene at a critical locations is given by

$$c = 1.75\left[erfc(0.6560) + e^{32.73} erfc(5.758)\right]$$

where

$$erfc(x) = \int_{\infty}^{x} e^{-z^2} dz$$

So in the above formula

$$erfc(0.6560) = \int_{\infty}^{0.6560} e^{-z^2} dz$$

Since e^{-z^2} decays rapidly as $z \to \infty$, we will approximate $erfc(0.6560) = \int_{5}^{0.6560} e^{-z^2} dz$

a) Use Richardson's rule to find the time required for 50% of the oxygen to be consumed. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.

b) Find the true error, E_t for part (a).

c) Find the absolute relative true error, $|\in_a|$ for part (a).

Solution

Table 2 Values obtained for Trapezoidal rule for

$$erfc(0.6560) = \int_{5}^{0.6560} e^{-z^2} dz$$

a)

n	Trapezoidal Rule
1	−1.4124
2	−0.70695
4	−0.40571
8	−0.33475

$$I_2 = -0.70695s$$

$$I_4 = -0.40571s$$

Using Richardson's extrapolation formula for Trapezoidal rule

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

and choosing *n=2*,

$$TV \approx I_4 + \frac{I_4 - I_2}{3} \quad = -0.40571 + \frac{-0.40571 - (-0.70695)}{3}$$

$$= -0.30530s$$

Solution (cont.)

b) The exact value of the above integral is cannot be found. For calculating the true error and relative true error, we assume the value obtained by adaptive numerical intregration using Maple as the exact value for calculating the true error and relative true error

$$erfc(0.6560) = \int_{5}^{0.6560} e^{-z^2} dz$$

$$= -0.31333$$

Hence

$$E_t = True\ Value - Approximate\ Value$$

$$= -0.31333 - (-0.30530)$$

$$= -0.0080295$$

Solution (cont.)

c) The absolute relative true error $\left|\in_t\right|$ would then be

$$\left|\in_t\right| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100 = \left|\frac{-0.0080295}{-0.31333}\right| \times 100 = 2.5627\%$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

Solution (cont.)

Table 2: The values obtained using Richardson's extrapolation formula for Trapezoidal rule for

$$erfc(0.6560) = \int_{5}^{0.6560} e^{-z^2} dz$$

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n	Trapezoidal Rule	$\lvert\in_t\rvert$ for Trapezoidal Rule	Richardson's Extrapolation	$\lvert\in_t\rvert$ for Richardson's Extrapolation
1	-1.4124	350.79	--	--
2	-0.70695	125.63	-0.47180	50.578
4	-0.40571	29.483	-0.30530	2.5627
8	-0.33475	6.8383	-0.31110	0.71156

Table 2: Richardson's Extrapolation Values

Romberg Integration

Romberg integration is same as Richardson's extrapolation formula as given previously. However, Romberg used a recursive algorithm for the extrapolation. Recall

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

This can alternately be written as

$$\left(I_{2n}\right)_R = I_{2n} + \frac{I_{2n} - I_n}{3} \quad = I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1}$$

Romberg Integration

Note that the variable TV is replaced by $(I_{2n})_R$ as the value obtained using Richardson's extrapolation formula. Note also that the sign $\approx$ is replaced by = sign.

Hence the estimate of the true value now is

$$TV \approx (I_{2n})_R + Ch^4$$

Where Ch^4 is an approximation of the true error.

Romberg Integration

Determine another integral value with further halving the step size (doubling the number of segments),

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3}$$

It follows from the two previous expressions that the true value TV can be written as

$$TV \approx (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{15}$$

$$= I_{4n} + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1}$$

Romberg Integration

A general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

The index *k* represents the order of extrapolation. k=1 represents the values obtained from the regular Trapezoidal rule, k=2 represents values obtained using the true estimate as $O(h^2)$. The index *j* represents the more and less accurate estimate of the integral.

Example 2

The concentration of benzene at a critical locations is given by

$$c = 1.75\left[erfc(0.6560) + e^{32.73}erfc(5.758)\right]$$

where

$$erfc(x) = \int_{\infty}^{x} e^{-z^2}dz$$

So in the above formula

$$erfc(0.6560) = \int_{\infty}^{0.6560} e^{-z^2}dz$$

Since e^{-z^2} decays rapidly as $z \to \infty$, we will approximate $erfc(0.6560) = \int_{5}^{0.6560} e^{-z^2}dz$

Use Romberg's rule to find *erfc*(0.6560).

Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given.

Solution

From Table 1, the needed values from original Trapezoidal rule are

$$I_{1,1} = -1.4124 \qquad I_{1,2} = -0.70695$$

$$I_{1,3} = -0.40571 \qquad I_{1,4} = -0.33475$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively.

Solution (cont.)

To get the first order extrapolation values,

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3}$$

$$= -0.70695 + \frac{-0.70695 - (-1.4124)}{3}$$

$$= -0.47180$$

Similarly,

$$I_{2,2} = I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3}$$

$$= -0.40571 + \frac{-0.40571 - (-0.70695)}{3}$$

$$= -0.30530$$

$$I_{2,3} = I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3}$$

$$= -0.33475 + \frac{-0.33475 - (-0.40571)}{3}$$

$$= -0.31110$$

Solution (cont.)

For the second order extrapolation values,

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}$$

$$= -0.30530 + \frac{-0.30530 - (-0.47180)}{15}$$

$$= -0.29420$$

Similarly,

$$I_{3,2} = I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15}$$

$$= -0.31110 + \frac{-0.31110 - (-0.30530)}{15}$$

$$= -0.31148$$

Solution (cont.)

For the third order extrapolation values,

$$I_{4,1} = I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63}$$

$$= -0.31148 + \frac{-0.31148 - (-0.29420)}{63}$$

$$= -0.31176$$

Table 3 shows these increased correct values in a tree graph.

Solution (cont.)

Table 3: Improved estimates of the integral value using Romberg Integration

		1st Order	2nd Order	3rd Order
1-segment	−1.4124			
		−0.47180		
			−0.29420	
2-segment	−0.70695			
				−0.31176
		−0.30530		
			−0.31148	
4-segment	−0.40571			
		−0.31110		
8-segment	−0.33475			

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/romberg_method.html

THE END

http://numericalmethods.eng.usf.edu