Euler Method

Civil Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

Euler Method

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Euler's Method

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

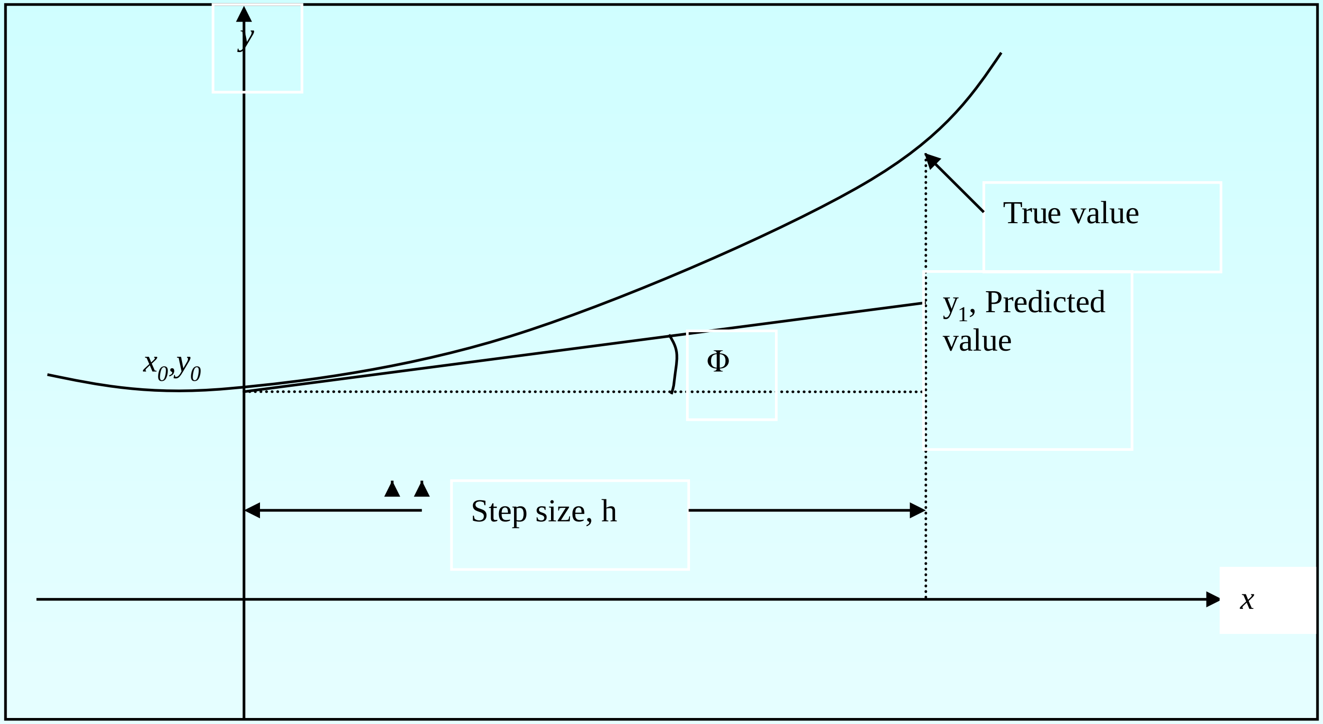

Figure 1 Graphical interpretation of the first step of Euler's method

Euler's Method

$$y_{i+1} = y_i + f\left(x_i, y_i\right)h$$

$$h = x_{i+1} - x_i$$

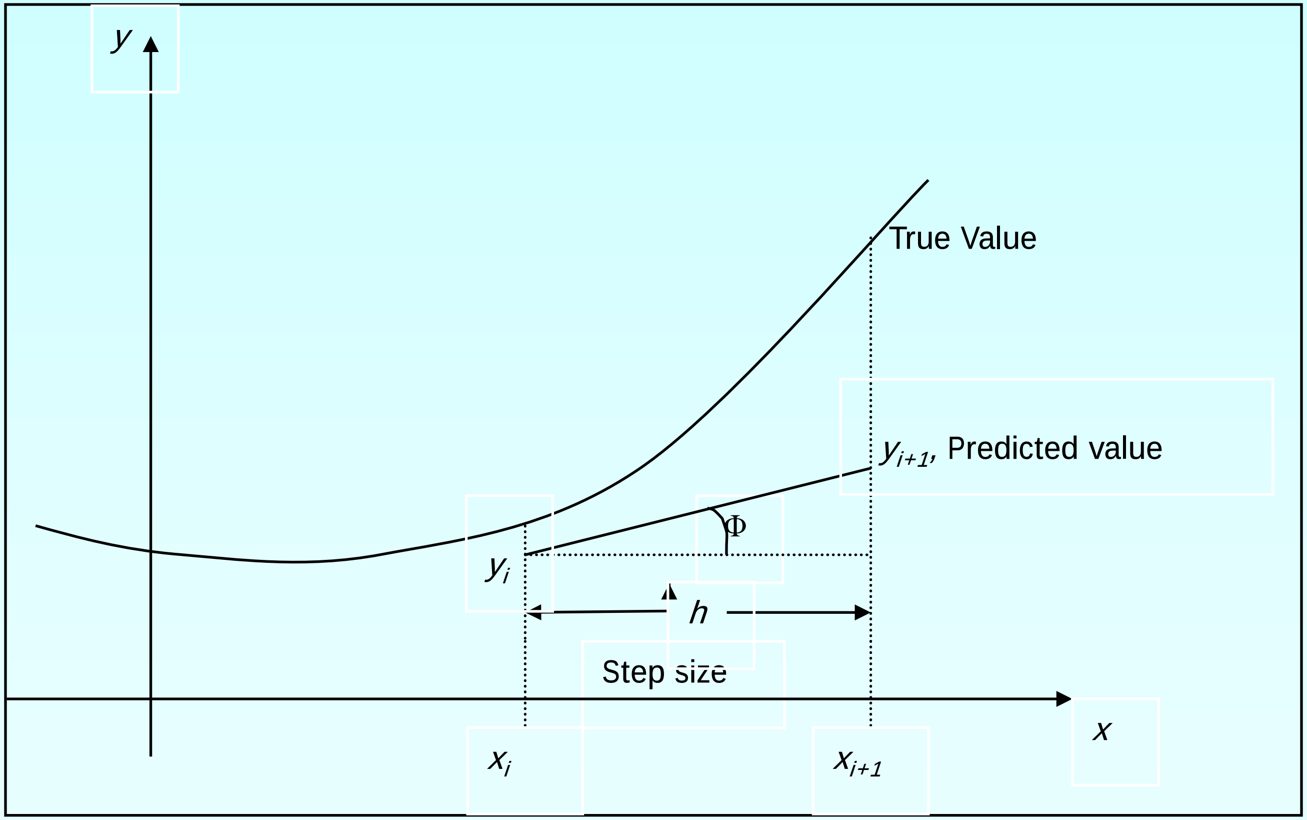

Figure 2. General graphical interpretation of Euler's method

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A polluted lake with an initial concentration of a bacteria is 10^7 parts/m^3, while the acceptable level is only 5×10^6 parts/m^3. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration *C* of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt}+0.06C=0, C(0)=10^7$$

Find the concentration of the pollutant after 7 weeks. Take a step size of 3.5 weeks.

$$\frac{dC}{dt}=-0.06C$$

$$f\left(t,C\right)=-0.06C$$

$$C_{i+1}=C_i+f\left(t_i,C_i\right)h$$

Solution

Step 1: For $i = 0,\ t_0 = 0,\ C_0 = 10^7$

$$
\begin{aligned}
C_1 &= C_0 + f(t_0, C_0)h \\
&= 10^7 + f(0, 10^7)3.5 \\
&= 10^7 + (-0.06(10^7))3.5 \\
&= 10^7 + (-6\times 10^5)3.5 \\
&= 7.9\times 10^6\ parts/m^3
\end{aligned}
$$

C_1 is the approximate concentration of bacteria at

$$t = t_1 = t_0 + h = 0 + 3.5 = 3.5 \text{ weeks}$$

$$C(3.5) \approx C_1 = 7.9\times 10^6 \text{ parts/m}^3$$

Solution Cont

Step 2: For $i=1,\ t_1=3.5,\ C_1=7.9\times10^6$

$$\begin{aligned}C_2 &= C_1 + f\left(t_1, C_1\right)h\\ &= 7.9\times10^6 + f\left(3.5,\ 7.9\times10^6\right)3.5\\ &= 7.9\times10^6 + \left(-0.06\left(7.9\times10^6\right)\right)3.5\\ &= 7.9\times10^6 + \left(-4.74\times10^5\right)3.5\\ &= 6.241\times10^6\ \text{parts/m}^3\end{aligned}$$

C_2 is the approximate concentration of bacteria at

$$t = t_2 = t_1 + h = 3.5 + 3.5 = 7\ \text{weeks}$$

$$C(7) \approx C_2 = 6.241\times10^6\ \text{parts/m}^3$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$C(t) = 1 \times 10^7 e^{\left(\frac{-3t}{50}\right)}$$

The solution to this nonlinear equation at t=7 weeks is

$$C(7) = 6.5705 \times 10^6 \text{ parts/m}^3$$

Comparison of Exact and Numerical Solutions

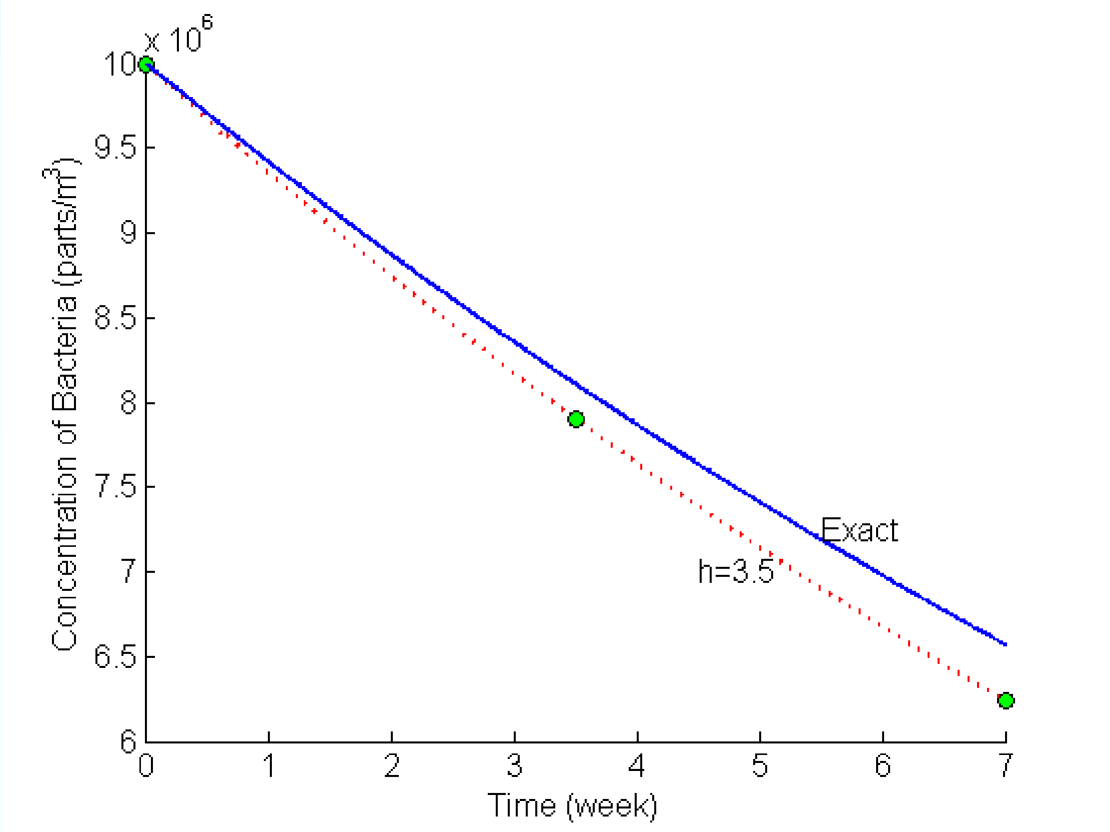

Figure 3. Comparing exact and Euler's method

Effect of step size

Table 1 Concentration of bacteria after 7 weeks as a function of step size

Step size h	$C(7)$	E_t	$\lvert\in_t\rvert\,\%$
7	$5.8\ \ 10^6$	770470	11.726
3.5	$6.241\ \ 10^6$	329470	5.0144
1.75	$6.4164\ \ 10^6$	154060	2.3447
0.875	$6.4959\ \ 10^6$	74652	1.1362
0.4375	$6.5337\ \ 10^6$	36763	0.55952

Comparison with exact results

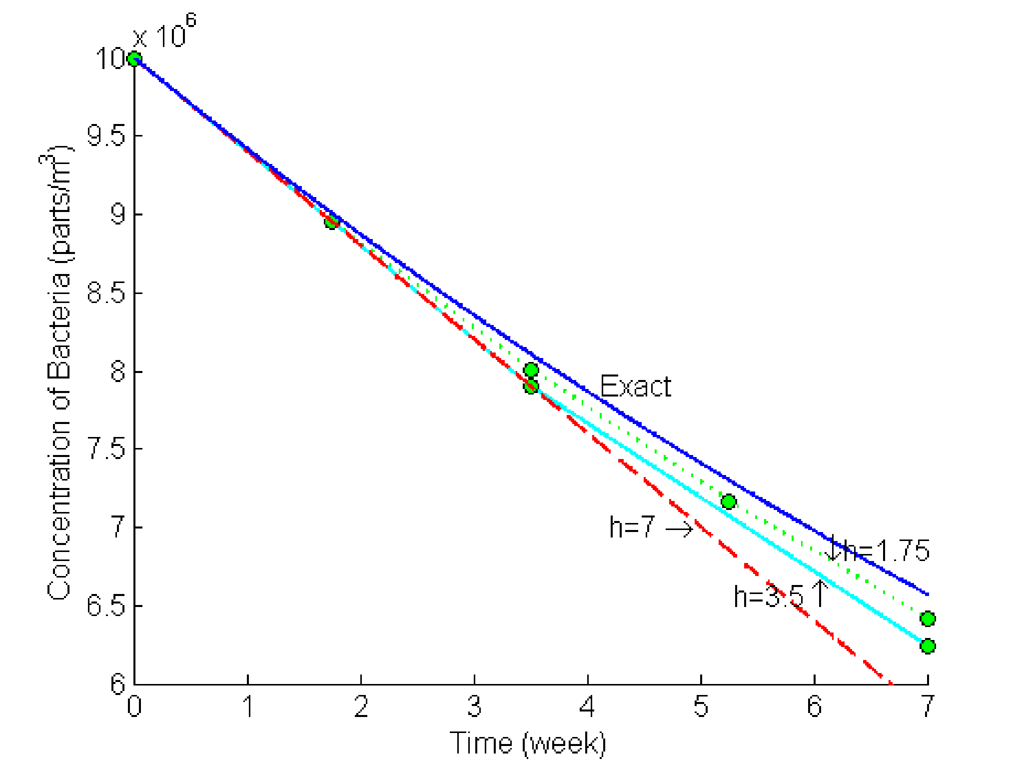

Figure 4. Comparison of Euler's method with exact solution for different step sizes

Effects of step size on Euler's Method

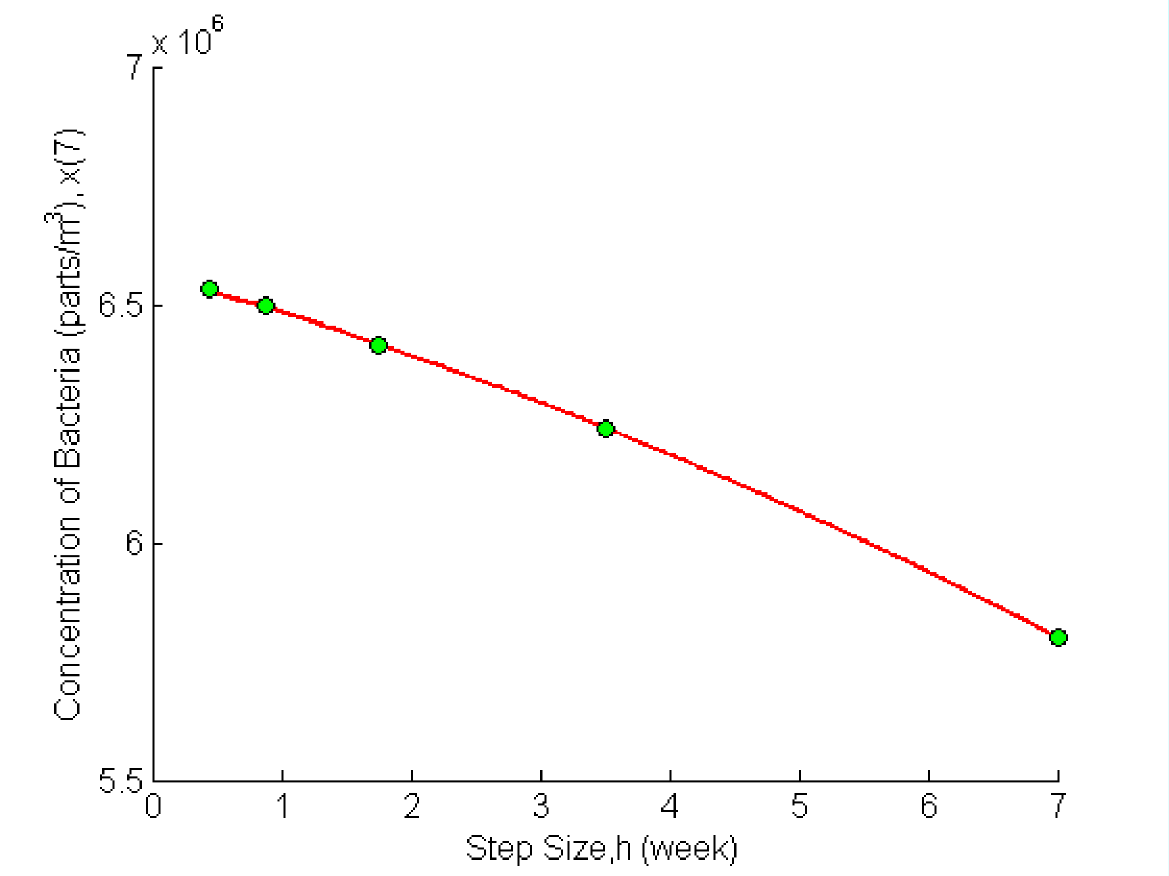

Figure 5. Effect of step size in Euler's method.

Errors in Euler's Method

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i, y_i} (x_{i+1} - x_i)^3 + \ldots$$

$$y_{i+1} = y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \ldots$$

As you can see the first two terms of the Taylor series

$y_{i+1} = y_i + f(x_i, y_i)h$ are the Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \ldots \qquad E_t \propto h^2$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/euler_method.html

THE END

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