Runge 4th Order Method

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Transforming Numerical Methods Education for STEM Undergraduates

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Runge-Kutta 4th Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 4th order method is given by

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f\left(x_i + h, y_i + k_3 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A polluted lake with an initial concentration of a bacteria is 10^7 parts/m^3, while the acceptable level is only 5×10^6 parts/m^3. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration *C* of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt} + 0.06C = 0, C(0) = 10^7$$

Find the concentration of the pollutant after 7 weeks. Take a step size of 3.5 weeks.

$$\frac{dC}{dt} = -0.06C$$

$$f(t, C) = -0.06C$$

$$C_{i+1} = C_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

Solution

Step 1: $i = 0,\ t_0 = 0,\ C_0 = 10^7\ parts/m^3$

$$k_1 = f(t_0, C_0) = f(0,\ 10^7) = -0.06(10^7) = -600000$$

$$k_2 = f\left(t_0 + \frac{1}{2}h,\ C_0 + \frac{1}{2}k_1 h\right) = f\left(0 + \frac{1}{2}3.5,\ 10^7 + \frac{1}{2}(-600000)3.5\right)$$
$$= f(1.75,\ 8950000) = -0.06(8950000) = -537000$$

$$k_3 = f\left(t_0 + \frac{1}{2}h,\ C_0 + \frac{1}{2}k_2 h\right) = f\left(0 + \frac{1}{2}3.5,\ 10^7 + \frac{1}{2}(-537000)3.5\right)$$
$$= f(1.75,\ 9060300) = -0.06(9060300) = -543620$$

$$k_4 = f(t_0 + h,\ C_0 + k_3 h) = f(0 + 3.5,\ 10^7 + (-543620)3.5)$$
$$= f(1.75,\ 8097300) = -0.06(8097300) = -485840$$

Solution Cont

$$C_1 = C_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= 10^7 + \frac{1}{6}\left(-600000 + 2\left(-537000\right) + 2\left(-543620\right) + \left(-485840\right)\right)3.5$$

$$= 10^7 + \frac{1}{6}\left(-3247100\right)3.5$$

$$= 8.1059 \times 10^6$$

C_1 is the approximate concentration of bacteria at

$$t = t_1 = t_0 + h = 0 + 3.5 = 3.5 \text{ weeks}$$

$$C\left(3.5\right) \approx C_1 = 8.1059 \times 10^6 \text{ parts/m}^3$$

Solution Cont

Step 2: $i = 1, \quad t_1 = 3.5, \quad C_1 = 8.1059 \times 10^6$

$$k_1 = f(t_1, C_1) = f(3.5,\ 8.1059 \times 10^6) = -0.06(8.1059 \times 10^6) = -486350$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, C_1 + \frac{1}{2}k_1 h\right) = f\left(3.5 + \frac{1}{2}(3.5),\ 8105900 + \frac{1}{2}(-486350)3.5\right)$$
$$= f(5.25,\ 7254800) = -0.06(7254800) = -435290$$

$$k_3 = f\left(t_1 + \frac{1}{2}h,\ C_1 + \frac{1}{2}k_2 h\right) = f\left(3.5 + \frac{1}{2}(3.5),\ 8105900 + \frac{1}{2}(-435290)3.5\right)$$
$$= f(5.25,\ 7344100) = -0.06(7344100) = -440648$$

$$k_4 = f(t_1 + h,\ C_1 + k_3 h) = f(3.5 + 3.5,\ 8105900 + (-440648)3.5)$$
$$= f(7,\ 6563600) = -0.06(6563600) = -393820$$

Solution Cont

$$C_2 = C_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 8105900 + \frac{1}{6}\left(-484350 + 2\left(-435290\right) + 2\left(-440648\right) + \left(-39820\right)\right)3.5$$

$$= 8105900 + \frac{1}{6}\left(-2632000\right)3.5$$

$$= 6.5705 \times 10^6 \ \text{parts/m}^3$$

C_2 is the approximate concentration of bacteria at

$$t_2 = t_1 + h = 3.5 + 3.5 = 7 weeks$$

$$C\left(7\right) \approx C_2 = 6.5705 \times 10^6 \ \text{parts/m}^3$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$C(t) = 1\times 10^{7} e^{\left(\frac{-3t}{50}\right)}$$

The solution to this nonlinear equation at t=7 weeks is

$$C(7) = 6.5705\times 10^{6}\ \text{parts/m}^{3}$$

Comparison with exact results

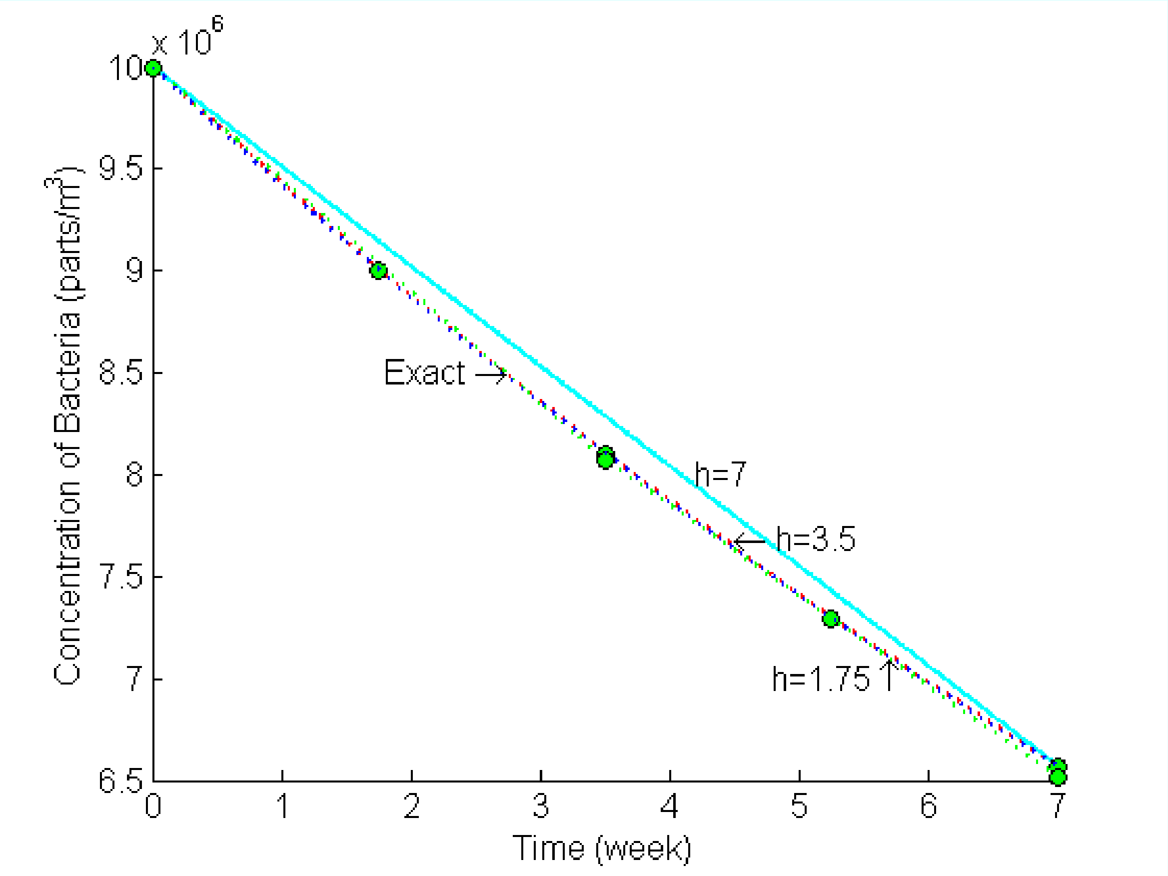

Figure 1. Comparison of Runge-Kutta 4th order method with exact solution

Effect of step size

Table 1 Value of concentration of bacteria at 3 minutes for different step sizes

Step size h	$C(7)$	E_t	$\lvert \in_t \rvert \%$
7	$6.5715\ \ 10^{6}$	-1017.2	0.015481
3.5	$6.5705\ \ 10^{6}$	-53.301	$8.1121\ \ 10^{-4}$
1.75	$6.5705\ \ 10^{6}$	-3.0512	$4.6438\ \ 10^{-5}$
0.875	$6.5705\ \ 10^{6}$	-0.18252	$2.7779\ \ 10^{-6}$
0.4375	$6.5705\ \ 10^{6}$	-0.011161	$1.6986\ \ 10^{-7}$

$C(7) = 6.5705 \times 10^{6}$ (exact)

Effects of step size on Runge-Kutta 4th Order Method

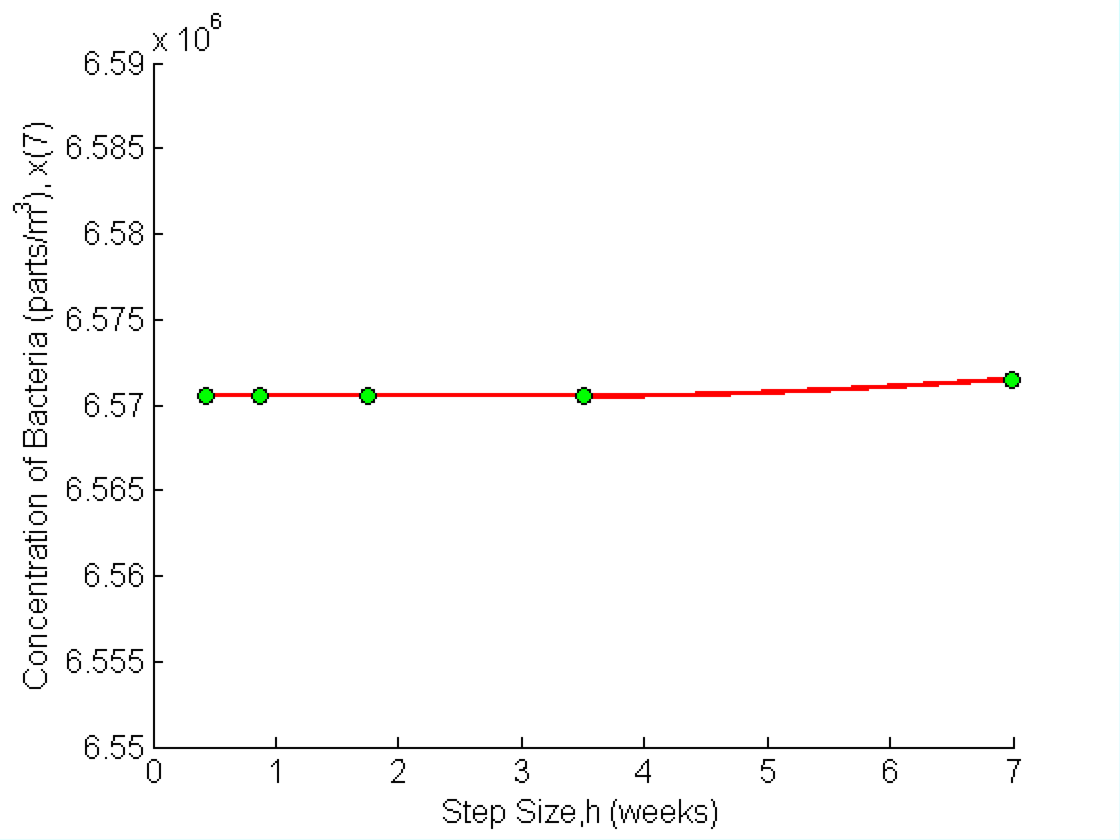

Figure 2. Effect of step size in Runge-Kutta 4th order method

Comparison of Euler and Runge-Kutta Methods

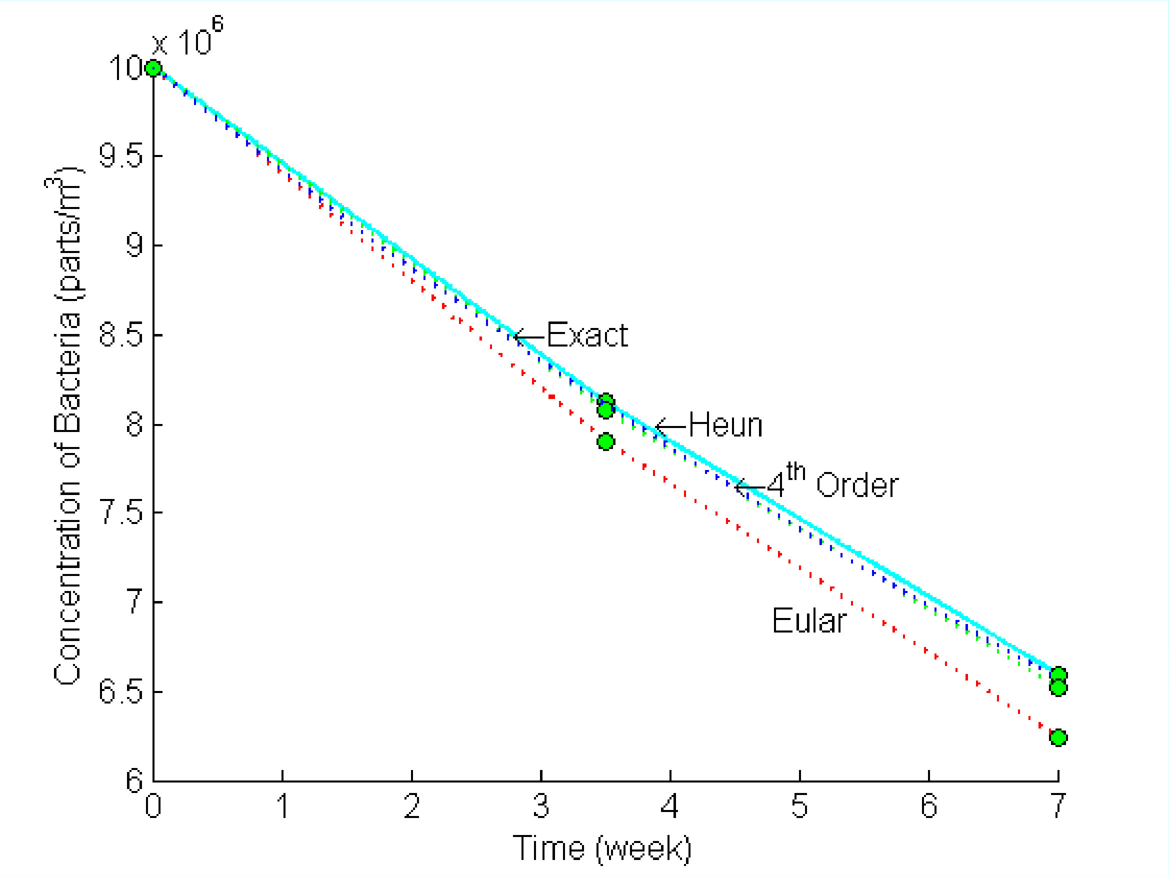

Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_4th_method.html

THE END

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