Chapter 03.00D

Physical Problem for Nonlinear Equations
Computer Engineering

Problem Statement

Many super computers do not have a unit to divide numbers. But why? Well, a divide operation in modern computers can take 20 to 25 clock cycles, and that is five times what it takes for multiplication [1]. Instead, a divide unit, based on numerically solving a nonlinear equation, is developed. This allows for a faster divide operation. This is how it works.
If you want to find the value of b/a where b and a are real numbers, one can look at

$$\frac{b}{a} = b * \frac{1}{a} = b * c$$

where

$$c = \frac{1}{a}$$

So, if one is able to find c, we only need to multiply b and c to find b/a. So how do we find c without the divide unit.
Equation

$$c = \frac{1}{a}$$

can be written as an equation

$$f(c) = ac - 1 = 0$$

If one is able to find the root of this equation without using a division, then we have the value of $1/a$.

Although we do not explain the numerical methods of solving nonlinear equations in this section of the notes, it becomes necessary to do so in this example. The Newton-Raphson method of solving a nonlinear equation is used in finding $1/a$. The Newton Raphson method of solving an equation $f(c) = 0$ is given by the iterative formula

$$c_{i+1} = c_i - \frac{f(c_i)}{f'(c_i)},$$

where

c_{i+1} is the new approximation of of the root of $f(c) = 0$ and

c_i is the previous approximation of the root of $f(c) = 0$.

What is the appropriate function to use to find the inverse of a?

a) Using

$$f(c) = ac - 1 = 0$$

gives

$$f'(c) = a$$

and the Newton-Raphson method formula gives

$$c_{i+1} = c_i - \frac{ac_i - 1}{a}$$

$$c_{i+1} = \frac{1}{a}$$

This is of no use as it involves division.

b) Using

$$f(c) = a - \frac{1}{c} = 0$$

gives

$$f'(c) = \frac{1}{c^2}$$

$$c_{i+1} = c_i - \frac{a - \dfrac{1}{c_i}}{\dfrac{1}{c_i^2}}$$

$$= c_i - c_i^2\left(a - \frac{1}{c_i}\right)$$

$$c_{i+1} = c_i(2 - c_i a)$$

This one is the acceptable iterative formula to find the inverse of a as it does not involve division.

Starting with an initial guess for the inverse of a, one can find newer approximations by using the above iterative formula. Each iteration requires two multiplications and one subtraction. However, the number of iterations required to find the inverse of a very much depends on the initial approximation. More accurate is the starting approximation, less number of iterations are required to find the inverse of a. Since the convergence of Newton Raphson method is quadratic, it may take up to six iterations to get an accurate reciprocal in double precision. By using look-up tables for the initial approximation, the number of iterations required can be reduced to two [2]. Also, the operation of $(2 - c_i a)$ may be carried in a fused multiply-subtract unit to further reduce the clock cycles needed for the computation.

References

1. Oberman, S.F., Flynn, M.J., "Division Algorithms and Implementations", IEEE Transactions on Computers, Vol. 46, No. 8, 1997.
2. Wong, D., Flynn, M.J., "Fat Division using accurate quotient approximations to reduce the number of iteratoimns", IEEE Transactions on Computers, Vol. 41, No. 8, 981-995, 1992.

Appendix A: Example of using Newton–Raphson method to find the inverse of a number.

Let us find $\dfrac{1}{2.5}$

The Newton –Raphson method formula is given by

$$c_{i+1} = c_i(2 - 2.5c_i)$$

Starting with estimate of inverse as $c_0 = 0.5$,

$$c_1 = 0.5[2 - 2.5(0.5)]$$
$$= 0.375$$

$$c_2 = 0.375[2 - 2.5(0.375)]$$
$$= 0.3984$$

$$c_3 = 0.3984[2 - 2.5(0.3984)]$$
$$= 0.3999$$

$$c_4 = 0.3999[2 - 2.5(0.3999)]$$
$$= 0.4000$$

It took four iterations to find the inverse of 2.5 correct up to 4 significant digits.

Topic	NONLINEAR EQUATION
Sub Topic	Physical Problem
Summary	For efficient design of the Cray supercomputer, it does not have a divide unit. It uses solution of a nonlinear equation to find the inverse of a number.
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