# Chapter 05.04
# Lagrangian Interpolation

*After reading this chapter, you should be able to:*

1. *derive Lagrangian method of interpolation,*
2. *solve problems using Lagrangian method of interpolation, and*
3. *use Lagrangian interpolants to find derivatives and integrals of discrete functions.*

**What is interpolation?**

Many times, data is given only at discrete points such as $(x_0, y_0)$, $(x_1, y_1)$, ......, $(x_{n-1}, y_{n-1})$, $(x_n, y_n)$. So, how then does one find the value of $y$ at any other value of $x$? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of $y$ at any other value of $x$. This is called *interpolation*.

Of course, if $x$ falls outside the range of $x$ for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

(A) evaluate,
(B) differentiate, and
(C) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order $n$ that passes through the $n+1$ data points. One of the methods used to find this polynomial is called the Lagrangian method of interpolation. Other methods include Newton's divided difference polynomial method and the direct method. We discuss the Lagrangian method in this chapter.

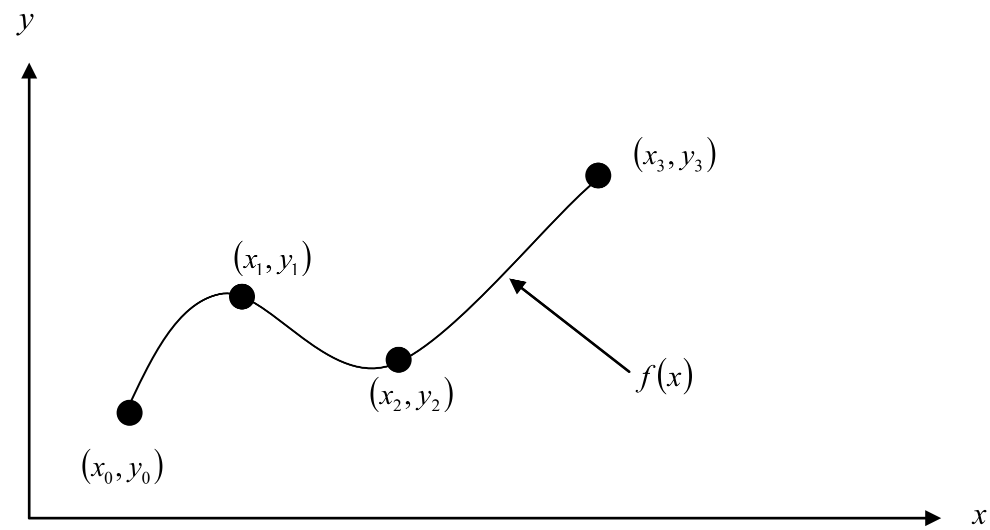


**Figure 1** Interpolation of discrete data.

The Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where $n$ in $f_n(x)$ stands for the $n^{\text{th}}$ order polynomial that approximates the function $y = f(x)$ given at $n+1$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j = i$ omitted. The application of Lagrangian interpolation will be clarified using an example.

**Example 1**

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a 15"×10" rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 1.

**Table 1** The coordinates of the holes on the plate.

| $x$ (in.) | $y$ (in.) |
|---|---|
| 2.00 | 7.2 |
| 4.25 | 7.1 |
| 5.25 | 6.0 |
| 7.81 | 5.0 |
| 9.20 | 3.5 |
| 10.60 | 5.0 |

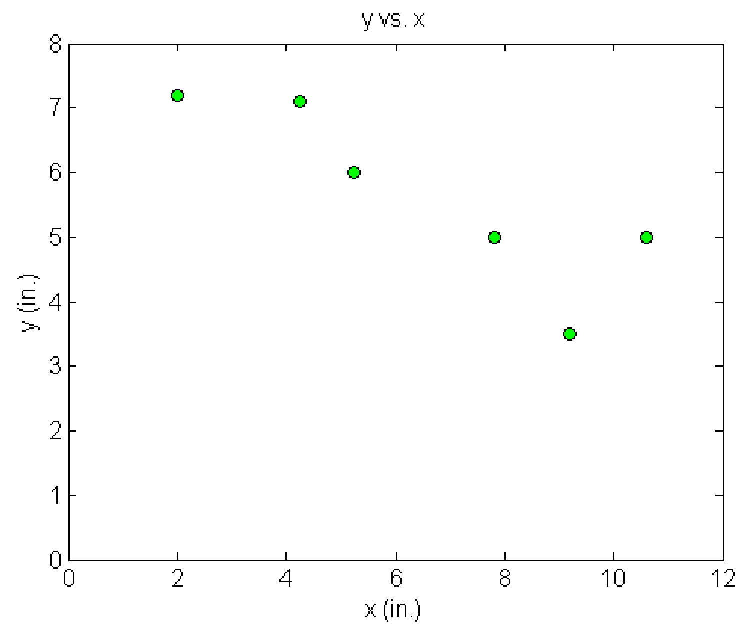


**Figure 2** Location of the holes on the rectangular plate.

If the laser is traversing from $x = 2$ to $x = 4.25$ in a linear path, what is the value of $y$ at $x = 4.00$ using the Lagrangian method and a first order polynomial?

**Solution**

For first order Lagrange polynomial interpolation (also called linear interpolation), we choose the value of $y$ as given by

$$\begin{aligned} y(x) &= \sum_{i=0}^{1} L_i(x)y(x_i) \\ &= L_0(x)y(x_0) + L_1(x)y(x_1) \end{aligned}$$

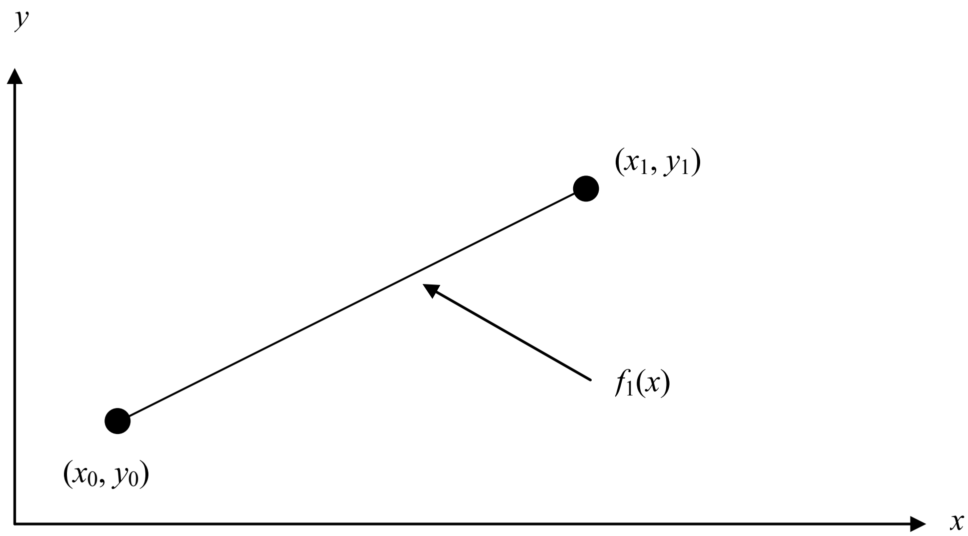


**Figure 3** Linear interpolation.

Since we want to find the value of $y$ at $x = 4.00$, using the two points $x_0 = 2.00$ and $x_1 = 4.25$, then

$$x_0 = 2.00,\ y(x_0) = 7.2$$
$$x_1 = 4.25,\ y(x_1) = 7.1$$

gives

$$L_0(x) = \prod_{\substack{j=0\\ j\neq 0}}^{1} \frac{x - x_j}{x_0 - x_j}$$
$$= \frac{x - x_1}{x_0 - x_1}$$
$$L_1(x) = \prod_{\substack{j=0\\ j\neq 1}}^{1} \frac{x - x_j}{x_1 - x_j}$$
$$= \frac{x - x_0}{x_1 - x_0}$$

Hence

$$y(x) = \frac{x - x_1}{x_0 - x_1} y(x_0) + \frac{x - x_0}{x_1 - x_0} y(x_1)$$
$$= \frac{x - 4.25}{2.00 - 4.25}(7.2) + \frac{x - 2.00}{4.25 - 2.00}(7.1),\ \ 2.00 \le x \le 4.25$$
$$y(4.00) = \frac{4.00 - 4.25}{2.00 - 4.25}(7.2) + \frac{4.00 - 2.00}{4.25 - 2.00}(7.1)$$
$$= 0.11111(7.2) + 0.88889(7.1)$$
$$= 7.1111 \text{ in.}$$

You can see that $L_0(x) = 0.11111$ and $L_1(x) = 0.88889$ are like weightages given to the values of $y$ at $x_0 = 2.00$ and $x_1 = 4.25$ to calculate the value of $y$ at $x = 4.00$.

**Example 2**

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a $15''\times 10''$ rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 2.

**Table 2** The coordinates of the holes on the plate.

| $x$ (in.) | $y$ (in.) |
|---|---|
| 2.00 | 7.2 |
| 4.25 | 7.1 |
| 5.25 | 6.0 |
| 7.81 | 5.0 |
| 9.20 | 3.5 |
| 10.60 | 5.0 |

If the laser is traversing from $x = 2.00$ to $x = 4.25$ to $x = 5.25$ in a quadratic path, what is the value of $y$ at $x = 4.00$ using a second order Lagrange polynomial? Find the absolute relative approximate error for the second order polynomial approximation.

**Solution**

For second order Lagrange polynomial interpolation (also called quadratic interpolation), we choose the value of $y$ given by

$$\begin{aligned} y(x) &= \sum_{i=0}^{2} L_i(x)y(x_i) \\ &= L_0(x)y(x_0) + L_1(x)y(x_1) + L_2(x)y(x_2) \end{aligned}$$

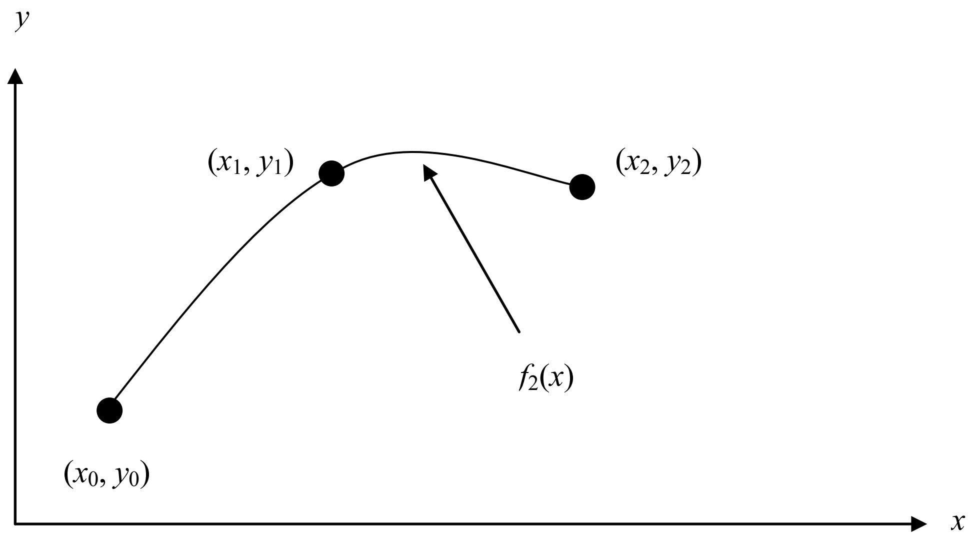


**Figure 4** Quadratic interpolation.

Since we want to find the value of $y$ at $x = 4.00$, using the three points as $x_0 = 2.00$, $x_1 = 4.25$ and $x_2 = 5.25$, then

$$x_0 = 2.00, \quad y(x_0) = 7.2$$
$$x_1 = 4.25, \quad y(x_1) = 7.1$$
$$x_2 = 5.25, \quad y(x_2) = 6.0$$

gives

$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^{2} \frac{x - x_j}{x_0 - x_j}$$
$$= \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)$$
$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^{2} \frac{x - x_j}{x_1 - x_j}$$
$$= \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)$$
$$L_2(x) = \prod_{\substack{j=0 \\ j \neq 2}}^{2} \frac{x - x_j}{x_2 - x_j}$$
$$= \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)$$

Hence

$$y(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)y(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)y(x_1) + \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)y(x_2),$$

$$x_0 \le x \le x_2$$

$$\begin{aligned} y(4.00) &= \frac{(4.00-4.25)(4.00-5.25)}{(2.00-4.25)(2.00-5.25)}(7.2) + \frac{(4.00-2.00)(4.00-5.25)}{(4.25-2.00)(4.25-5.25)}(7.1) \\ &\quad + \frac{(4.00-2.00)(4.00-4.25)}{(5.25-2.00)(5.25-4.25)}(6.0) \\ &= (0.042735)(7.2) + (1.1111)(7.1) + (-0.15385)(6.0) \\ &= 7.2735 \text{ in.} \end{aligned}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$\begin{aligned} |\in_a| &= \left|\frac{7.2735 - 7.1111}{7.2735}\right| \times 100 \\ &= 2.2327\% \end{aligned}$$

**Example 3**

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a $15'' \times 10''$ rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 3.

**Table 3** The coordinates of the holes on the plate.

| $x$ (in.) | $y$ (in.) |
|---|---|
| 2.00 | 7.2 |
| 4.25 | 7.1 |
| 5.25 | 6.0 |
| 7.81 | 5.0 |
| 9.20 | 3.5 |
| 10.60 | 5.0 |

Find the path traversed through the six points using a fifth order Lagrange polynomial.

**Solution**

For fifth order Lagrange polynomial interpolation (also called quintic interpolation), we choose the value of $y$ given by

$$\begin{aligned} y(x) &= \sum_{i=0}^{5} L_i(x)y(x_i) \\ &= L_0(x)y(x_0) + L_1(x)y(x_1) + L_2(x)y(x_2) \\ &\quad + L_3(x)y(x_3) + L_4(x)y(x_4) + L_5(x)y(x_5) \end{aligned}$$

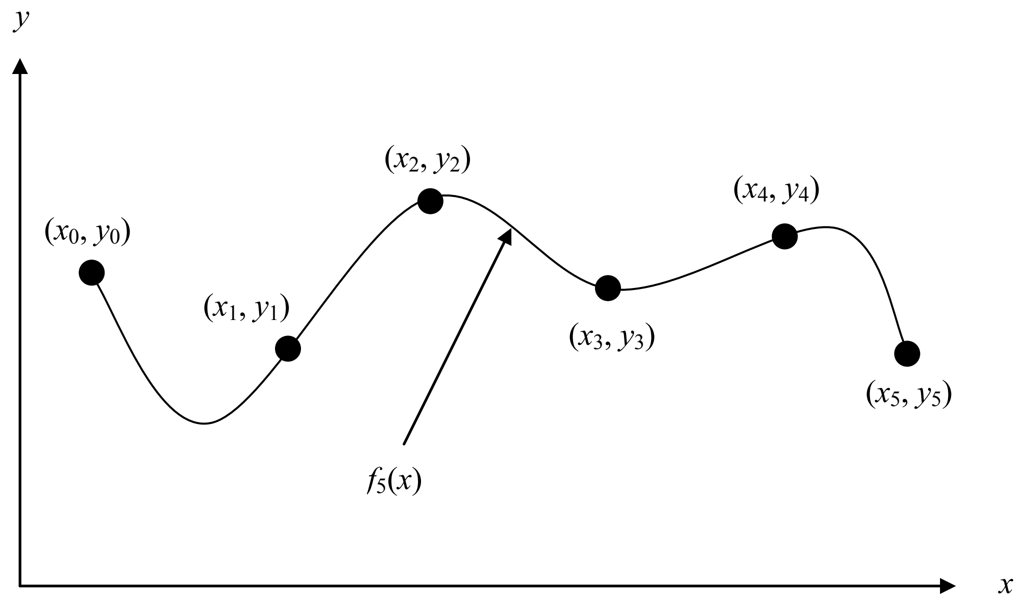


**Figure 5** 5th order polynomial interpolation.

Using the six points,

$$x_0 = 2.00, \quad y(x_0) = 7.2$$
$$x_1 = 4.25, \quad y(x_1) = 7.1$$
$$x_2 = 5.25, \quad y(x_2) = 6.0$$
$$x_3 = 7.81, \quad y(x_3) = 5.0$$
$$x_4 = 9.20, \quad y(x_4) = 3.5$$
$$x_5 = 10.60, \quad y(x_5) = 5.0$$

gives

$$L_0(x) = \prod_{\substack{j=0 \\ j\neq 0}}^{5} \frac{x - x_j}{x_0 - x_j}$$

$$= \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)\left(\frac{x - x_3}{x_0 - x_3}\right)\left(\frac{x - x_4}{x_0 - x_4}\right)\left(\frac{x - x_5}{x_0 - x_5}\right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j\neq 1}}^{5} \frac{x - x_j}{x_1 - x_j}$$

$$= \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right)\left(\frac{x - x_4}{x_1 - x_4}\right)\left(\frac{x - x_5}{x_1 - x_5}\right)$$

$$L_2(x) = \prod_{\substack{j=0 \\ j\neq 2}}^{5} \frac{x - x_j}{x_2 - x_j}$$

$$= \left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right)\left(\frac{x-x_4}{x_2-x_4}\right)\left(\frac{x-x_5}{x_2-x_5}\right)$$

$$L_3(x) = \prod_{\substack{j=0 \\ j\neq 3}}^{5} \frac{x-x_j}{x_3-x_j}$$

$$= \left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)\left(\frac{x-x_4}{x_3-x_4}\right)\left(\frac{x-x_5}{x_3-x_5}\right)$$

$$L_4(x) = \prod_{\substack{j=0 \\ j\neq 4}}^{5} \frac{x-x_j}{x_4-x_j}$$

$$= \left(\frac{x-x_0}{x_4-x_0}\right)\left(\frac{x-x_1}{x_4-x_1}\right)\left(\frac{x-x_2}{x_4-x_2}\right)\left(\frac{x-x_3}{x_4-x_3}\right)\left(\frac{x-x_5}{x_4-x_5}\right)$$

$$L_5(x) = \prod_{\substack{j=0 \\ j\neq 5}}^{5} \frac{x-x_j}{x_5-x_j}$$

$$= \left(\frac{x-x_0}{x_5-x_0}\right)\left(\frac{x-x_1}{x_5-x_1}\right)\left(\frac{x-x_2}{x_5-x_2}\right)\left(\frac{x-x_3}{x_5-x_3}\right)\left(\frac{x-x_4}{x_5-x_4}\right)$$

$$y(x) = \left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)\left(\frac{x-x_3}{x_0-x_3}\right)\left(\frac{x-x_4}{x_0-x_4}\right)\left(\frac{x-x_5}{x_0-x_5}\right)y(x_0)$$

$$+ \left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)\left(\frac{x-x_4}{x_1-x_4}\right)\left(\frac{x-x_5}{x_1-x_5}\right)y(x_1)$$

$$+ \left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right)\left(\frac{x-x_4}{x_2-x_4}\right)\left(\frac{x-x_5}{x_2-x_5}\right)y(x_2)$$

$$+ \left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)\left(\frac{x-x_4}{x_3-x_4}\right)\left(\frac{x-x_5}{x_3-x_5}\right)y(x_3)$$

$$+ \left(\frac{x-x_0}{x_4-x_0}\right)\left(\frac{x-x_1}{x_4-x_1}\right)\left(\frac{x-x_2}{x_4-x_2}\right)\left(\frac{x-x_3}{x_4-x_3}\right)\left(\frac{x-x_5}{x_4-x_5}\right)y(x_4)$$

$$+ \left(\frac{x-x_0}{x_5-x_0}\right)\left(\frac{x-x_1}{x_5-x_1}\right)\left(\frac{x-x_2}{x_5-x_2}\right)\left(\frac{x-x_3}{x_5-x_3}\right)\left(\frac{x-x_4}{x_5-x_4}\right)y(x_5)$$

$$
\begin{aligned}
&= \frac{(x-4.25)(x-5.25)(x-7.81)(x-9.20)(x-10.60)}{(2.00-4.25)(2.00-5.25)(2.00-7.81)(2.00-9.20)(2.00-10.60)}(7.2)\\
&+ \frac{(x-2.00)(x-5.25)(x-7.81)(x-9.20)(x-10.60)}{(4.25-2.00)(4.25-5.25)(4.25-7.81)(4.25-9.20)(4.25-10.60)}(7.1)\\
&+ \frac{(x-2.00)(x-4.25)(x-7.81)(x-9.20)(x-10.60)}{(5.25-2.00)(5.25-4.25)(5.25-7.81)(5.25-9.20)(5.25-10.60)}(6.0)\\
&+ \frac{(x-2.00)(x-4.25)(x-5.25)(x-9.20)(x-10.60)}{(7.81-2.00)(7.81-4.25)(7.81-5.25)(7.81-9.20)(7.81-10.60)}(5.0)\\
&+ \frac{(x-2.00)(x-4.25)(x-5.25)(x-7.81)(x-10.60)}{(9.20-2.00)(9.20-4.25)(9.20-5.25)(9.20-7.81)(9.20-10.60)}(3.5)\\
&+ \frac{(x-2.00)(x-4.25)(x-5.25)(x-7.81)(x-9.20)}{(10.60-2.00)(10.60-4.25)(10.60-5.25)(10.60-7.81)(10.60-9.20)}(5.0)\\
&= \frac{x^5-37.11x^4+536.77x^3-3773.2x^2+12862x-16994}{-365.38}\\
&+ \frac{x^5-34.86x^4+462.83x^3-2879.7x^2+8169.5x-7997.1}{35.461}\\
&+ \frac{x^5-33.86x^4+433.22x^3-2572.3x^2+6903.5x-6473.9}{-29.304}\\
&+ \frac{x^5-31.3x^4+366.53x^3-1984.1x^2+4912.4x-4351.8}{41.069}\\
&+ \frac{x^5-29.91x^4+335.81x^3-1757.2x^2+4241.6x-3694.3}{-78.273}\\
&+ \frac{x^5-28.51x^4+308.78x^3-1573.7x^2+3727.5x-3206.4}{228.24}
\end{aligned}
$$

$$
y(x) = -30.898+41.344x-15.855x^2+2.7862x^3-0.23091x^4+0.0072923x^5, \quad 2 \le x \le 10.6
$$

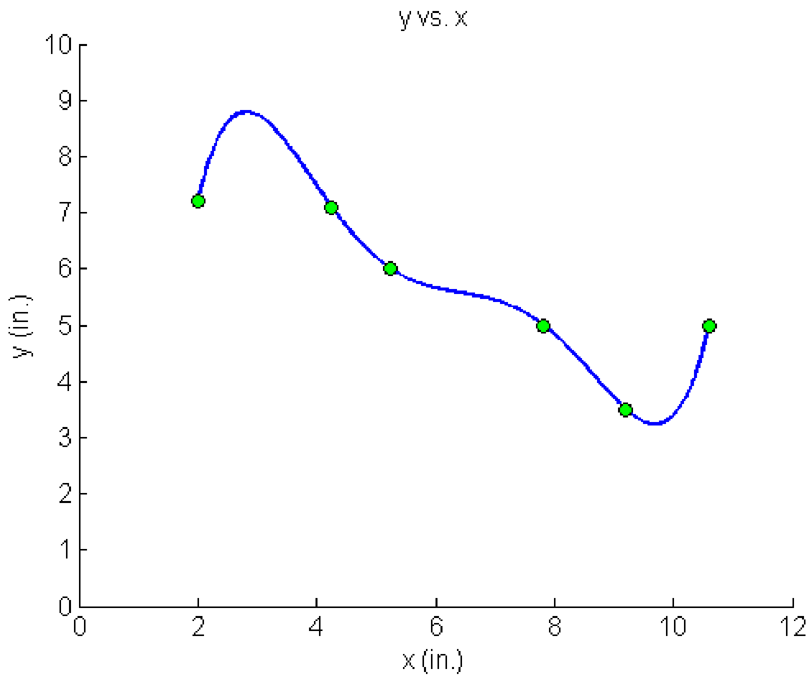


**Figure 6** Fifth order polynomial to traverse points of robot path (using Lagrangian method of interpolation).

| INTERPOLATION | |
|---|---|
| Topic | Lagrange Interpolation |
| Summary | Textbook notes on the Lagrangian method of interpolation |
| Major | Computer Engineering |
| Authors | Autar Kaw, Michael Keteltas |
| Last Revised | November 11, 2012 |
| Web Site | http://numericalmethods.eng.usf.edu |