Chapter 08.04
Runge-Kutta 4th Order Method for Ordinary Differential Equations

After reading this chapter, you should be able to

1. *develop Runge-Kutta 4th order method for solving ordinary differential equations,*
2. *find the effect size of step size has on the solution,*
3. *know the formulas for other versions of the Runge-Kutta 4th order method*

What is the Runge-Kutta 4th order method?

Runge-Kutta 4th order method is a numerical technique used to solve ordinary differential equation of the form

$$\frac{dy}{dx} = f(x,y), y(0) = y_0$$

So only first order ordinary differential equations can be solved by using the Runge-Kutta 4th order method. In other sections, we have discussed how Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x,y),\ \ y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$
$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example 2

Rewrite

$$e^{y}\frac{dy}{dx} + x^2y^2 = 2\sin(3x),\ y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y),\ y(0) = y_0 \text{ form.}$$

Solution

$$e^{y}\frac{dy}{dx} + x^2y^2 = 2\sin(3x),\ y(0) = 5$$
$$\frac{dy}{dx} = \frac{2\sin(3x) - x^2y^2}{e^{y}},\ y(0) = 5$$

In this case

$$f(x, y) = \frac{2\sin(3x) - x^2y^2}{e^{y}}$$

The Runge-Kutta 4th order method is based on the following

$$y_{i+1} = y_i + (a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4)h \tag{1}$$

where knowing the value of $y = y_i$ at x_i, we can find the value of $y = y_{i+1}$ at x_{i+1}, and

$$h = x_{i+1} - x_i$$

Equation (1) is equated to the first five terms of Taylor series

$$y_{i+1} = y_i + \frac{dy}{dx}\Big|_{x_i, y_i}(x_{i+1} - x_i) + \frac{1}{2!}\frac{d^2y}{dx^2}\Big|_{x_i, y_i}(x_{i+1} - x_i)^2 + \frac{1}{3!}\frac{d^3y}{dx^3}\Big|_{x_i, y_i}(x_{i+1} - x_i)^3 + \frac{1}{4!}\frac{d^4y}{dx^4}\Big|_{x_i, y_i}(x_{i+1} - x_i)^4 \tag{2}$$

Knowing that $\frac{dy}{dx} = f(x, y)$ and $x_{i+1} - x_i = h$

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2 + \frac{1}{3!}f''(x_i, y_i)h^3 + \frac{1}{4!}f'''(x_i, y_i)h^4 \tag{3}$$

Based on equating Equation (2) and Equation (3), one of the popular solutions used is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \tag{4}$$

$$k_1 = f(x_i, y_i) \tag{5a}$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) \tag{5b}$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right) \tag{5c}$$

$$k_4 = f(x_i + h, y_i + k_3 h) \tag{5d}$$

Example 3

A rectifier-based power supply requires a capacitor to temporarily store power when the rectified waveform from the AC source drops below the target voltage. To properly size this capacitor a first-order ordinary differential equation must be solved. For a particular power supply, with a capacitor of $150\ \mu\text{F}$, the ordinary differential equation to be solved is

$$\frac{dv(t)}{dt} = \frac{1}{150\times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(t))| - 2 - v(t)}{0.04}, 0\right)\right\}$$

$$v(0) = 0$$

Using the Runge-Kutta 4th order method, find voltage across the capacitor at $t = 0.00004\,\text{s}$. Use step size $h = 0.00002\,\text{s}$.

Solution

$$\frac{dv}{dt} = \frac{1}{150\times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(t))| - 2 - v}{0.04}, 0\right)\right\}$$

$$f(t, v) = \frac{1}{150\times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(t))| - 2 - v}{0.04}, 0\right)\right\}$$

$$v_{i+1} = v_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

For $i = 0$, $t_0 = 0$, $v_0 = 0$

$$\begin{aligned} k_1 &= f(t_0, v_0) \\ &= f(0,0) \\ &= \frac{1}{150\times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0))| - 2 - (0)}{0.04}, 0\right)\right\} \\ &= \frac{1}{150\times 10^{-6}}\{-0.1 + \max(400, 0)\} \\ &= \frac{1}{150\times 10^{-6}}\{-0.1 + 400\} \\ &= 2.6660\times 10^{6} \end{aligned}$$

$$\begin{aligned}
k_2 &= f\left(t_0 + \frac{1}{2}h, v_0 + \frac{1}{2}k_1 h\right) \\
&= f\left(0 + \frac{1}{2}(0.00002), 0 + \frac{1}{2}\left(2.6660 \times 10^6\right)0.00002\right) \\
&= f(0.00001, 26.660) \\
&= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0.00001))| - 2 - (26.660)}{0.04}, 0\right)\right\} \\
&= \frac{1}{150 \times 10^{-6}}\{-0.1 + \max(-266.50, 0)\} \\
&= \frac{1}{150 \times 10^{-6}}\{-0.1 - 266.50\} \\
&= -666.67
\end{aligned}$$

$$\begin{aligned}
k_3 &= f\left(t_0 + \frac{1}{2}h, v_0 + \frac{1}{2}k_2 h\right) \\
&= f\left(0 + \frac{1}{2}(0.00002), 0 + \frac{1}{2}(-666.67)0.00002\right) \\
&= f(0.00001, -0.0066667) \\
&= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0.00001))| - 2 - (-0.0066667)}{0.04}, 0\right)\right\} \\
&= \frac{1}{150 \times 10^{-6}}\{-0.1 + \max(400.16, 0)\} \\
&= \frac{1}{150 \times 10^{-6}}\{-0.1 + 400.16\} \\
&= 2.6671 \times 10^6
\end{aligned}$$

$$\begin{aligned}
k_4 &= f(t_0 + h, v_0 + k_3 h) \\
&= f\left(0 + 0.00002, 0 + \left(2.6671 \times 10^6\right)0.00002\right) \\
&= f(0.00002, 53.342) \\
&= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0.00002))| - 2 - (53.342)}{0.04}, 0\right)\right\} \\
&= \frac{1}{150 \times 10^{-6}}\{-0.1 + \max(-933.56, 0)\} \\
&= \frac{1}{150 \times 10^{-6}}\{-0.1 + 0\} \\
&= -666.67
\end{aligned}$$

$$\begin{aligned}
v_1 &= v_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\
&= 0 + \frac{1}{6}\left(2.6660 \times 10^6 + 2(-666.67) + 2\left(2.6671 \times 10^6\right) + (-666.67)\right)0.00002
\end{aligned}$$

$$= 0 + \frac{1}{6}\left(7.9982 \times 10^{6}\right)0.00002$$
$$= 26.661 \text{ V}$$

v_1 is the approximate voltage at

$$t = t_1 = t_0 + h = 0 + 0.00002 = 0.00002$$
$$v(0.00002) \approx v_1 = 26.661 \text{V}$$

For $i = 1, t_1 = 0.00002, v_1 = 26.661$

$$k_1 = f(t_1, v_1)$$
$$= f(0.00002, 26.661)$$
$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0.00002))\right| - 2 - (26.661)}{0.04}, 0\right)\right\}$$
$$= \frac{1}{150 \times 10^{-6}}\{-0.1 + \max(-266.51, 0)\}$$
$$= \frac{1}{150 \times 10^{-6}}\{-0.1 + 0\}$$
$$= -666.67$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, v_1 + \frac{1}{2}k_1 h\right)$$
$$= f\left(0.00002 + \frac{1}{2}(0.00002), 26.661 + \frac{1}{2}(-666.67)0.00002\right)$$
$$= f(0.00003, 26.654)$$
$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0.00003))\right| - 2 - (26.654)}{0.04}, 0\right)\right\}$$
$$= \frac{1}{150 \times 10^{-6}}\{-0.1 + \max(-266.35, 0)\}$$
$$= \frac{1}{150 \times 10^{-6}}\{-0.1 + 0\}$$
$$= -666.67$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, v_1 + \frac{1}{2}k_2 h\right)$$
$$= f\left(0.00002 + \frac{1}{2}(0.00002), 26.661 + \frac{1}{2}(-666.67)0.00002\right)$$
$$= f(0.00003, 26.654)$$
$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0.00003))\right| - 2 - (26.654)}{0.04}, 0\right)\right\}$$
$$= \frac{1}{150 \times 10^{-6}}\{-0.1 + \max(-266.35, 0)\}$$

$$= \frac{1}{150\times10^{-6}}\{-0.1+0\}$$
$$= -666.67$$

$$k_4 = f(t_1 + h, v_1 + k_3 h)$$
$$= f(0.00002 + (0.00002), 26.661 + (-666.67)0.00002)$$
$$= f(0.00003, 26.647)$$

$$= \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{|18\cos(120\pi(0.00003))| - 2 - (26.634)}{0.04}, 0\right)\right\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+\max(-265.87, 0)\}$$
$$= \frac{1}{150\times10^{-6}}\{-0.1+0\}$$
$$= -666.67$$

$$v_2 = v_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$
$$= 26.661 + \frac{1}{6}(-666.67 + 2(-666.67) + 2(-666.67) + (-666.67))0.00002$$
$$= 26.661 + \frac{1}{6}(-4000.0)0.00002$$
$$= 26.647\,\text{V}$$

v_2 is the approximate voltage at $t = t_2$

$$t_2 = t_1 + h = 0.00002 + 0.00002 = 0.00004\,\text{s}$$
$$v(.00004) \approx v_2 = 26.647\,\text{V}$$

Figure 1 compares the exact solution of $v(0.00004) = 15.974\,\text{V}$ with the numerical solution using Runge-Kutta 4th order method step size of $h = 0.00002\,\text{s}$.

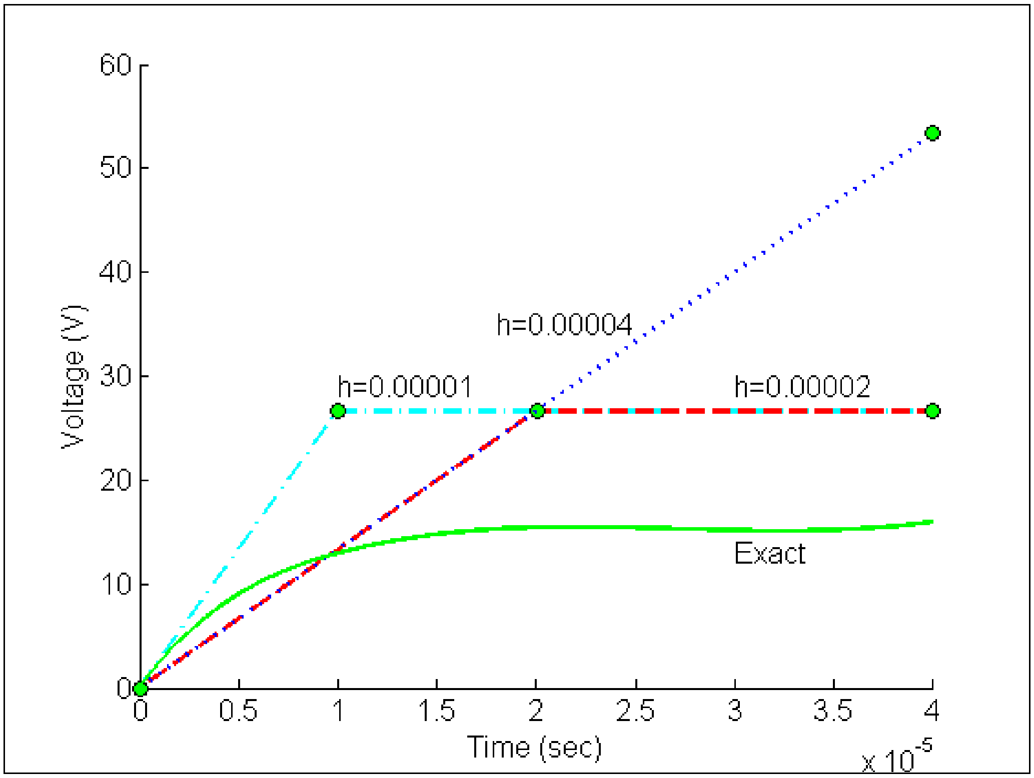

Figure 1 Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.

Table 1 and Figure 2 shows the effect of step size on the value of the calculated temperature at $t = 0.00004\,\text{s}$.

Table 1 Value of voltage at time, $t = 0.00004\,\text{s}$ for different step sizes.

Step size, h	$v(0.00004)$	E_t	$\lvert\in_t\rvert\,\%$
0.00004	53.335	−37.361	233.89
0.00002	26.647	−10.673	66.817
0.00001	15.986	−0.012299	0.076996
0.000005	15.975	−0.00050402	0.0031552
0.0000025	15.976	−0.0015916	0.0099639

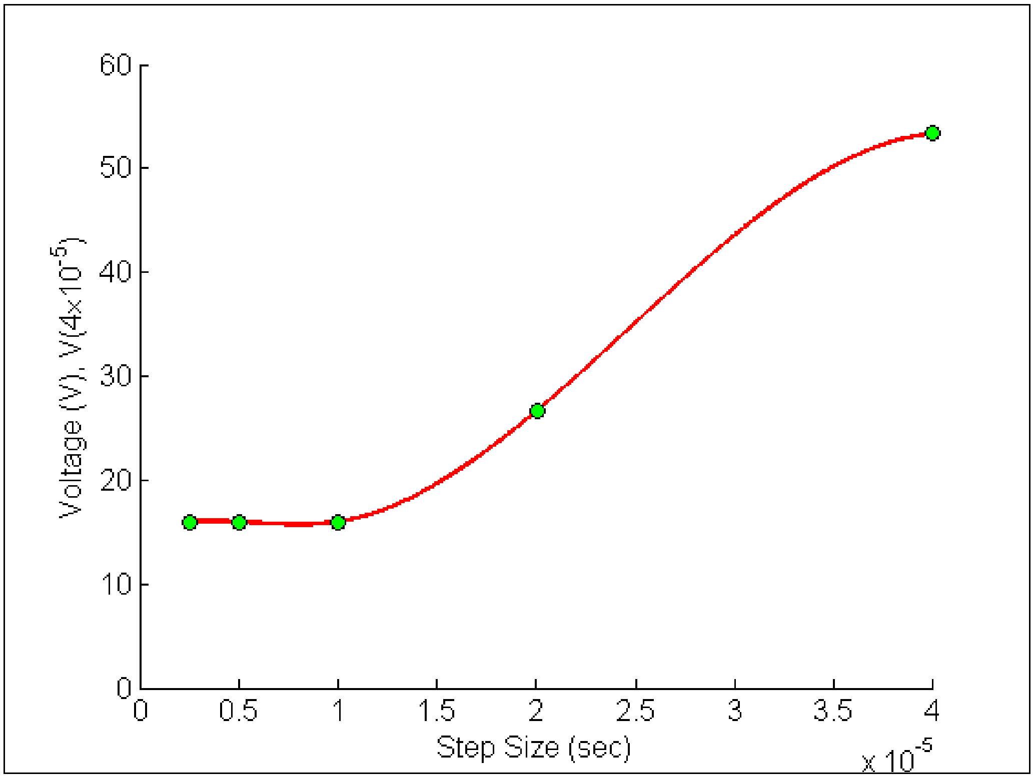

Figure 2 Effect of step size in Runge-Kutta 4th order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method), and Runge-Kutta 4th order method.

The formula described in this chapter was developed by Runge. This formula is same as Simpson's 1/3 rule, if $f(x,y)$ were only a function of x. There are other versions of the 4th order method just like there are several versions of the second order methods. The formula developed by Kutta is

$$y_{i+1} = y_i + \frac{1}{8}(k_1 + 3k_2 + 3k_3 + k_4)h \tag{6}$$

where

$$k_1 = f(x_i, y_i) \tag{7a}$$

$$k_2 = f\left(x_i + \frac{1}{3}h, y_i + \frac{1}{3}hk_1\right) \tag{7b}$$

$$k_3 = f\left(x_i + \frac{2}{3}h, y_i - \frac{1}{3}hk_1 + hk_2\right) \tag{7c}$$

$$k_4 = f(x_i + h, y_i + hk_1 - hk_2 + hk_3) \tag{7d}$$

This formula is the same as the Simpson's 3/8 rule, if $f(x,y)$ is only a function of x.

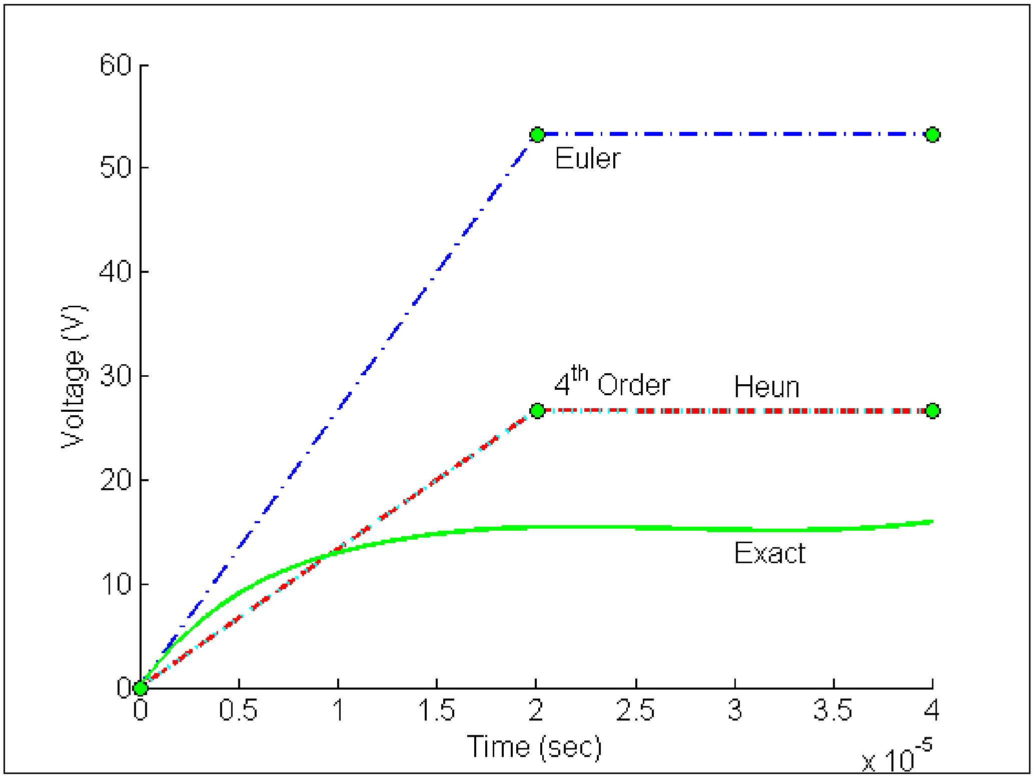

Figure 3 Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Runge-Kutta 4th order method
Summary	Textbook notes on the Runge-Kutta 4th order method for solving ordinary differential equations.
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