Chapter 01.06
Propagation of Errors

If a calculation is made with numbers that are not exact, then the calculation itself will have an error. How do the errors in each individual number propagate through the calculations. Let's look at the concept via some examples.

Example 1

Find the bounds for the propagation error in adding two numbers. For example if one is calculating $X+Y$ where

$$X = 1.5 \pm 0.05,$$
$$Y = 3.4 \pm 0.04 \text{ .}$$

Solution

By looking at the numbers, the maximum possible value of X and Y are

$$X = 1.55 \text{ and } Y = 3.44$$

Hence

$$X + Y = 1.55 + 3.44 = 4.99$$

is the maximum value of $X+Y$.

The minimum possible value of X and Y are

$$X = 1.45 \text{ and } Y = 3.36 .$$

Hence

$$X + Y = 1.45 + 3.36$$
$$= 4.81$$

is the minimum value of $X+Y$.

Hence

$$4.81 \le X + Y \le 4.99.$$

One can find similar intervals of the bound for the other arithmetic operations of $X-Y, X*Y,$ and X/Y. What if the evaluations we are making are function evaluations instead? How do we find the value of the propagation error in such cases.

If f is a function of several variables $X_1, X_2, X_3, \ldots\ldots., X_{n-1}, X_n$, then the maximum possible value of the error in f is

$$\Delta f \approx \left|\frac{\partial f}{\partial X_1}\Delta X_1\right| + \left|\frac{\partial f}{\partial X_2}\Delta X_2\right| + \ldots\ldots. + \left|\frac{\partial f}{\partial X_{n-1}}\Delta X_{n-1}\right| + \left|\frac{\partial f}{\partial X_n}\Delta X_n\right|$$

Example 2

The strain in an axial member of a square cross-section is given by

$$\in=\frac{F}{h^2E}$$

where

F =axial force in the member, N
h = length or width of the cross-section, m
E =Young's modulus, Pa

Given

$F = 72 \pm 0.9$ N
$h = 4 \pm 0.1$ mm
$E = 70 \pm 1.5$ GPa

Find the maximum possible error in the measured strain.

Solution

$$\in=\frac{72}{(4\times10^{-3})^2(70\times10^9)}$$
$$= 64.286\times10^{-6}$$
$$= 64.286\mu$$

$$\Delta\in=\left|\frac{\partial\in}{\partial F}\Delta F\right|+\left|\frac{\partial\in}{\partial h}\Delta h\right|+\left|\frac{\partial\in}{\partial E}\Delta E\right|$$

$$\frac{\partial\in}{\partial F}=\frac{1}{h^2E}$$

$$\frac{\partial\in}{\partial h}=-\frac{2F}{h^3E}$$

$$\frac{\partial\in}{\partial E}=-\frac{F}{h^2E^2}$$

$$\Delta E=\left|\frac{1}{h^2E}\Delta F\right|+\left|\frac{2F}{h^3E}\Delta h\right|+\left|\frac{F}{h^2E^2}\Delta E\right|$$

$$=\left|\frac{1}{(4\times10^{-3})^2(70\times10^9)}\times0.9\right|+\left|\frac{2\times72}{(4\times10^{-3})^3(70\times10^9)}\times0.0001\right|$$
$$+\left|\frac{72}{(4\times10^{-3})^2(70\times10^9)^2}\times1.5\times10^9\right|$$

$$= 8.0357\times10^{-7}+3.2143\times10^{-6}+1.3776\times10^{-6}$$
$$= 5.3955\times10^{-6}$$
$$= 5.3955\mu$$

Hence

$$\in=(64.286\mu\pm5.3955\mu)$$

implying that the axial strain, $\in$ is between 58.8905μ and 69.6815μ

Example 3

Subtraction of numbers that are nearly equal can create unwanted inaccuracies. Using the formula for error propagation, show that this is true.

Solution

Let

$$z = x - y$$

Then

$$\begin{aligned}|\Delta z| &= \left|\frac{\partial z}{\partial x}\Delta x\right| + \left|\frac{\partial z}{\partial y}\Delta y\right| \\ &= |(1)\Delta x| + |(-1)\Delta y| \\ &= |\Delta x| + |\Delta y|\end{aligned}$$

So the absolute relative change is

$$\left|\frac{\Delta z}{z}\right| = \frac{|\Delta x| + |\Delta y|}{|x - y|}$$

As x and y become close to each other, the denominator becomes small and hence create large relative errors.

For example if

$$x = 2 \pm 0.001$$

$$y = 2.003 \pm 0.001$$

$$\begin{aligned}\left|\frac{\Delta z}{z}\right| &= \frac{|0.001| + |0.001|}{|\,2 - 2.003\,|} \\ &= 0.6667 \\ &= 66.67\%\end{aligned}$$

INTRODUCTION TO NUMERICAL METHODS	
Topic	Propagation of Errors
Summary	Textbook notes on how errors propagate in arithmetic and function evaluations
Major	All Majors of Engineering
Authors	Autar Kaw
Last Revised	December 7, 2008
Web Site	http://numericalmethods.eng.usf.edu