Newton-Raphson Method

Electrical Engineering Majors

Authors: Autar Kaw, Jai Paul

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Transforming Numerical Methods Education for STEM Undergraduates

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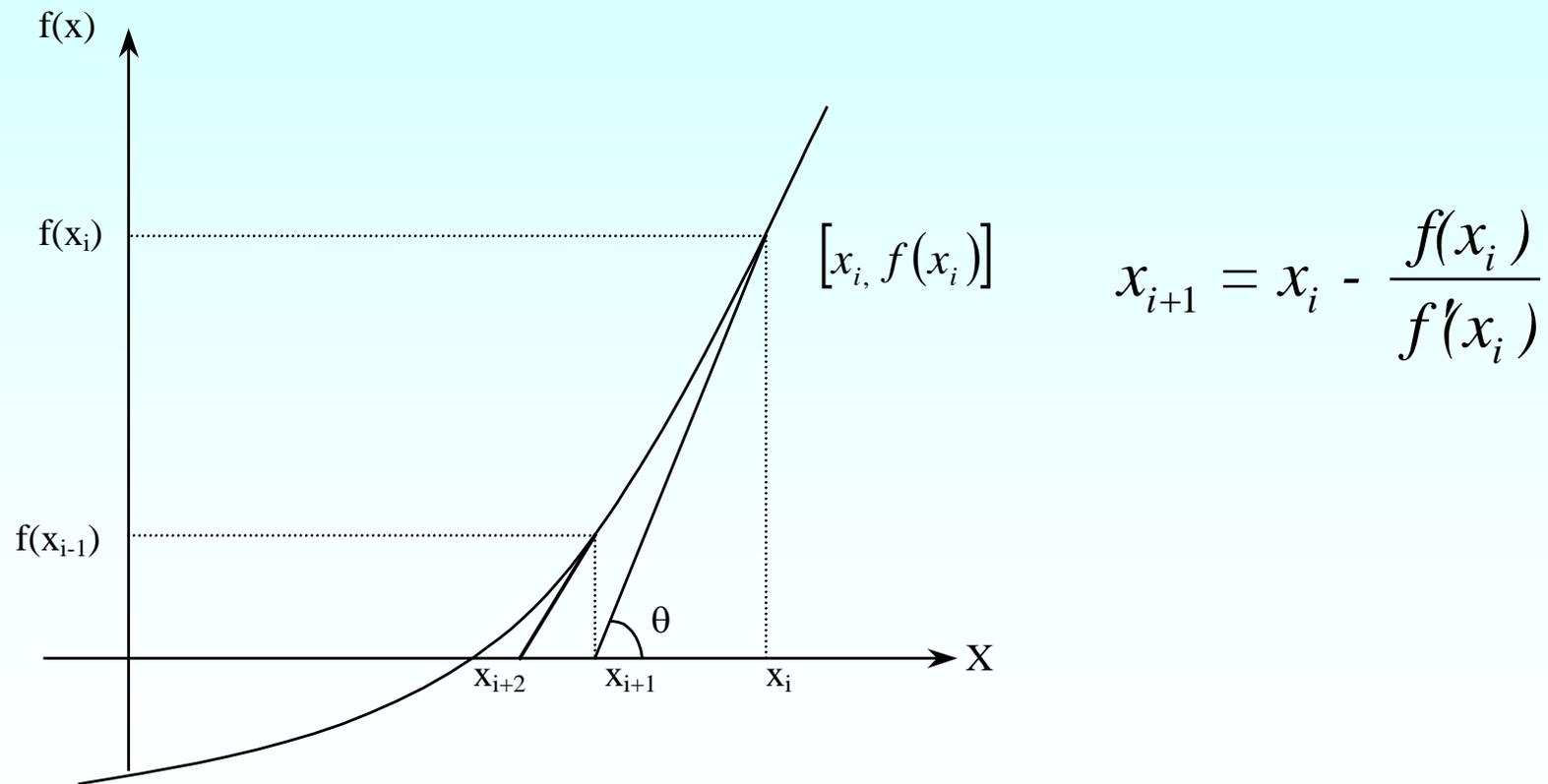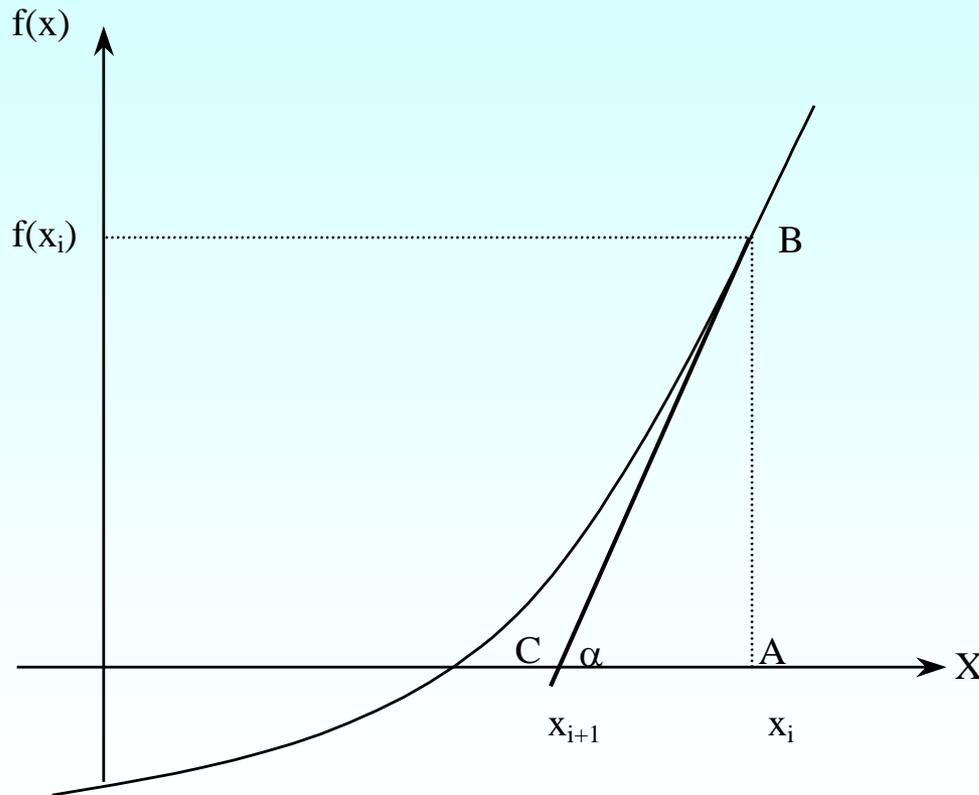$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 1 Geometrical illustration of the Newton-Raphson method.

Derivation

$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 2 Derivation of the Newton-Raphson method.

Algorithm for Newton-Raphson Method

Step 1

Evaluate $f'(x)$ symbolically.

Step 2

Use an initial guess of the root, x_i, to estimate the new value of the root, x_{i+1}, as

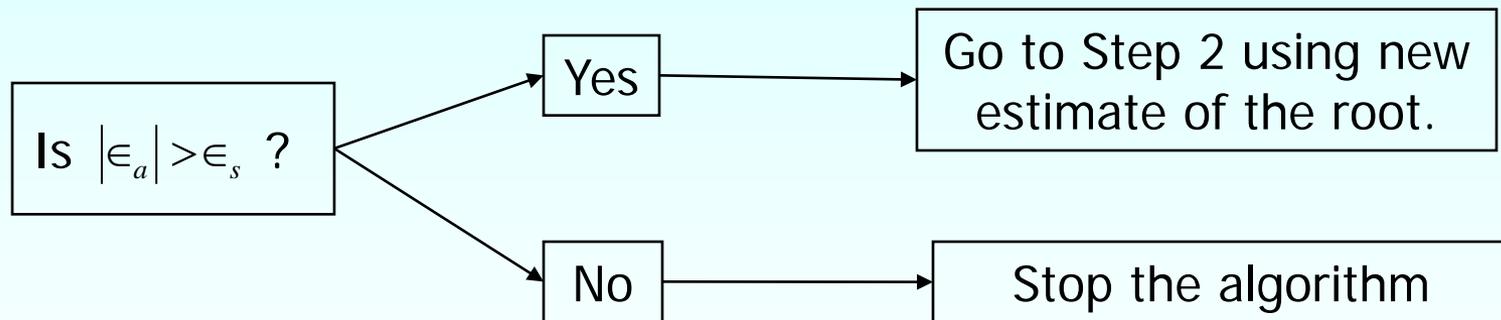$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Step 3

Find the absolute relative approximate error $\left| \in_a \right|$ as

$$\left| \in_a \right| = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100$$

Step 4

Compare the absolute relative approximate error with the pre-specified relative error tolerance $\in_s$.

Is $\left|\in_a\right| > \in_s$?

Yes → Go to Step 2 using new estimate of the root.

No → Stop the algorithm

Also, check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user.

Example 1

Thermistors are temperature-measuring devices based on the principle that the thermistor material exhibits a change in electrical resistance with a change in temperature. By measuring the resistance of the thermistor material, one can then determine the temperature.

For a 10K3A Betatherm thermistor, the relationship between the resistance, R, of the thermistor and the temperature is given by

Figure 3 A typical thermistor.

$$\frac{1}{T} = 1.129241 \times 10^{-3} + 2.341077 \times 10^{-4} \ln(R) + 8.775468 \times 10^{-8} \{\ln(R)\}^3$$
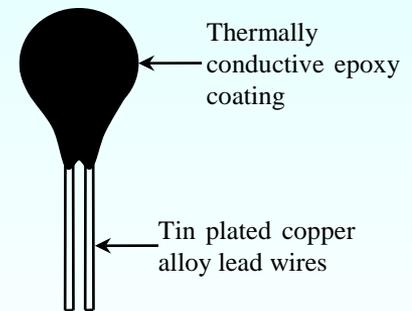
where T is in Kelvin and R is in ohms.

Example 1 Cont.

For the thermistor, error of no more than ±0.01ºC is acceptable. To find the range of the resistance that is within this acceptable limit at 19ºC, we need to solve

$$\frac{1}{19.01+273.15}=1.129241\times10^{-3}+2.341077\times10^{-4}\ln(R)+8.775468\times10^{-8}\{\ln(R)\}^3$$

and

$$\frac{1}{18.99+273.15}=1.129241\times10^{-3}+2.341077\times10^{-4}\ln(R)+8.775468\times10^{-8}\{\ln(R)\}^3$$

Use the Newton-Raphson method of finding roots of equations to find the resistance R at 18.99ºC.

a) Conduct three iterations to estimate the root of the above equation.

b) Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.
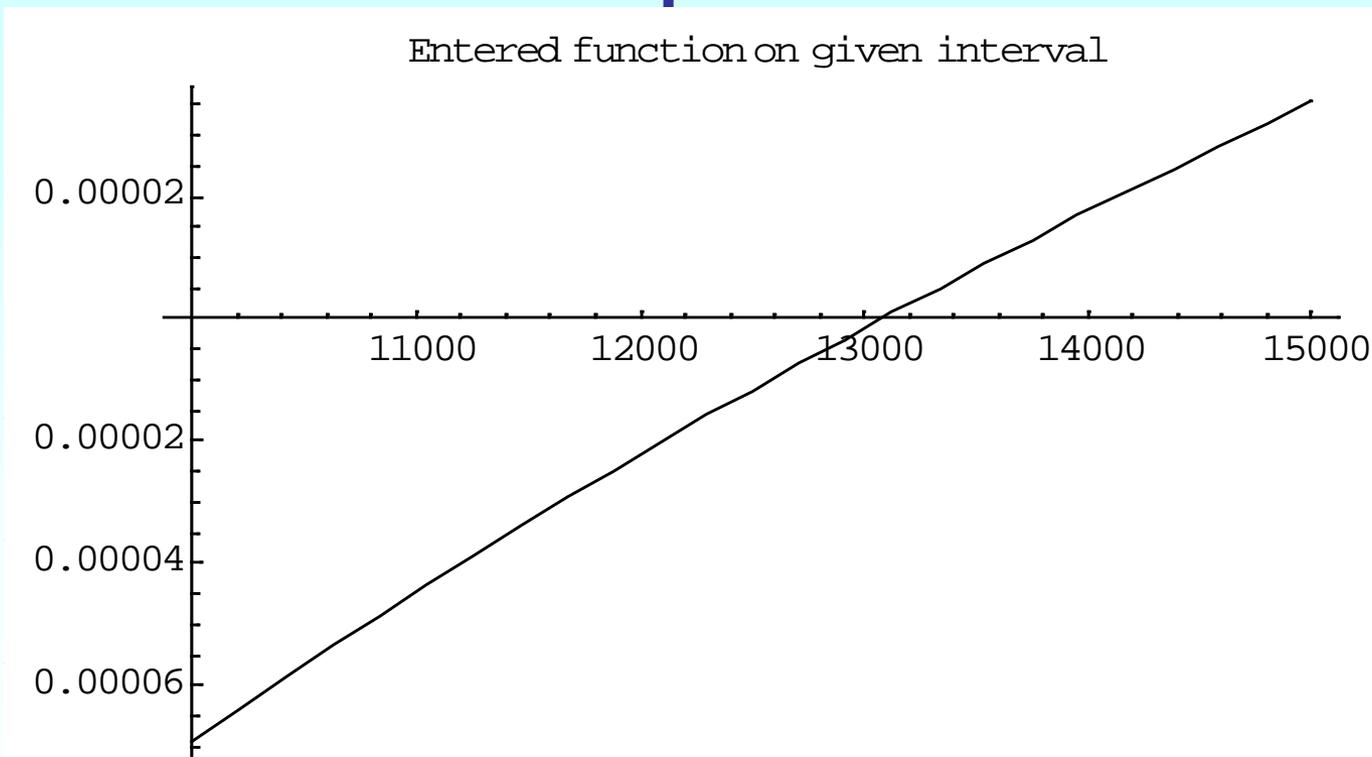
Example 1 Cont.

Entered function on given interval

0.00002

11000   12000   13000   14000   15000

0.00002

0.00004

0.00006

Figure 4 Graph of the function f(R).

$$f(R) = 2.341077 \times 10^{-4} \ln(R) + 8.775468 \times 10^{-8} \{\ln(R)\}^3 - 2.293775 \times 10^{-3}$$

Example 1 Cont.

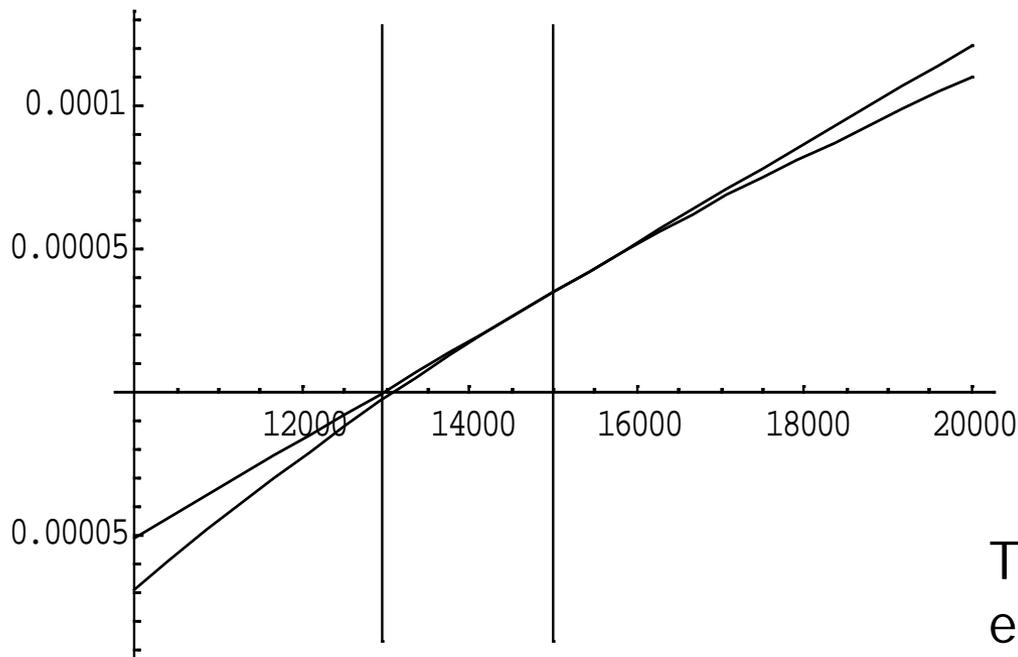Entered function on given interval with guess and estimated root

Figure 5 Graph of the estimate of the root after Iteration 1.

Initial guess: $R_0 = 15000$

<u>Iteration 1</u>

The estimate of the root is

$$R_1 = R_0 - \frac{f(R_0)}{f'(R_0)}$$

$$R_1 = 15000 - \frac{3.5383 \times 10^{-5}}{1.7230 \times 10^{-8}}$$

$$= 12946$$

The absolute relative approximate error is

$$\left| \in_a \right| = \left| \frac{12946 - 15000}{12946} \right| \times 100$$

$$= 15.862\%$$

The number of significant digits at least correct is 0.

Example 1 Cont.

Entered function on given interval with guess and estimated root

Figure 6 Graph of the estimate of the root after Iteration 2.

<u>Iteration 2</u>
The estimate of the root is

$$R_2 = R_1 - \frac{f(R_1)}{f'(\mathbf{R}_1)}$$

$$R_2 = 12946 - \frac{-2.6140 \times 10^{-6}}{1.9906 \times 10^{-8}}$$

$$= 13078$$

The absolute relative approximate error is

$$|\in_a| = \left|\frac{13078 - 12946}{13078}\right| \times 100$$

$$= 1.0041\%$$

The number of significant digits at least correct is 1.

Example 1 Cont.

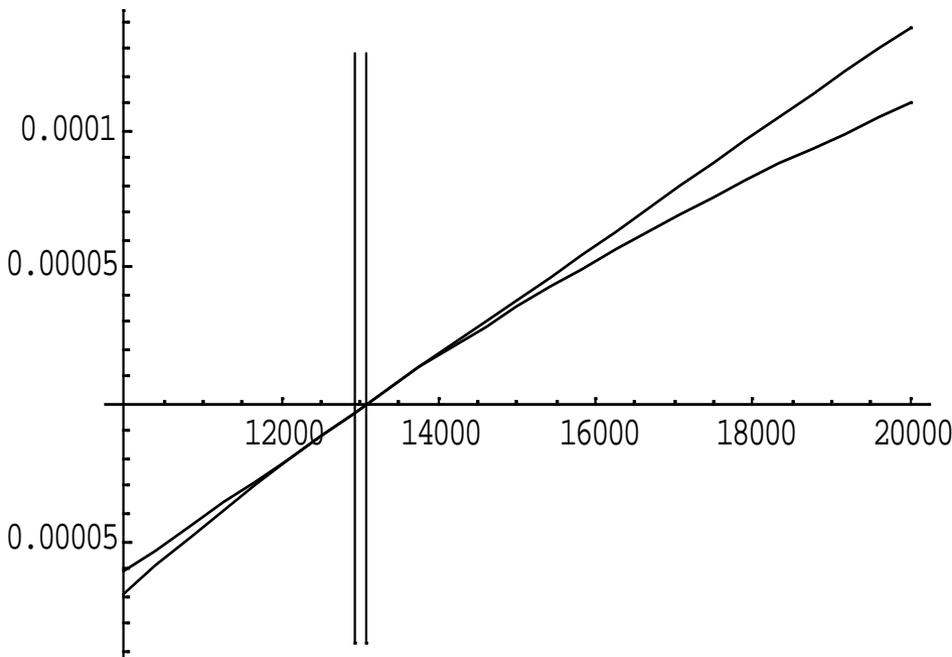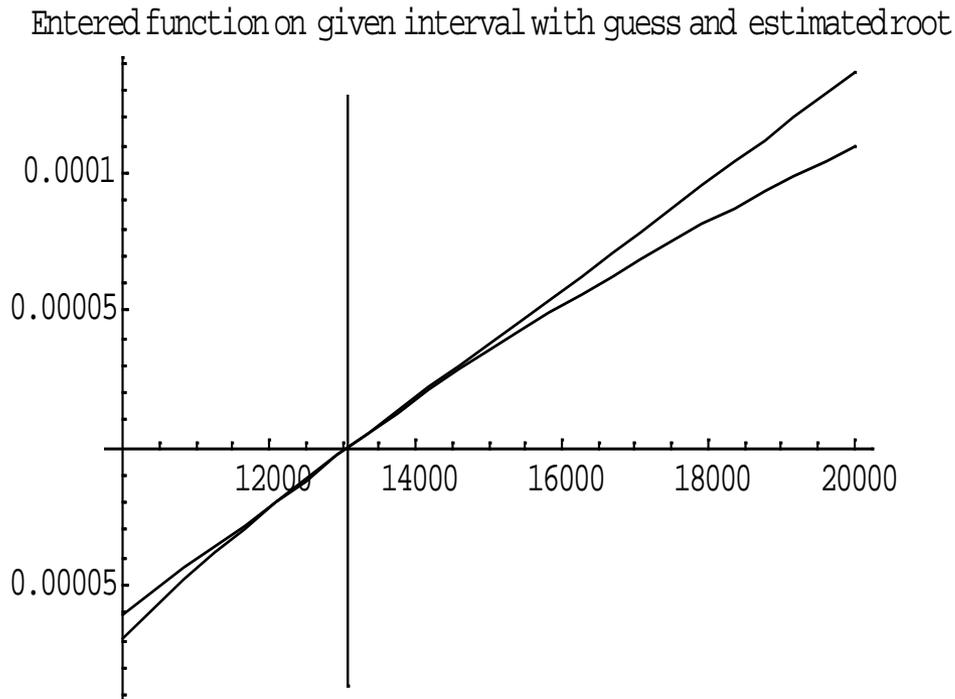Entered function on given interval with guess and estimated root

Figure 7 Graph of the estimate of the root after Iteration 3.

<u>Iteration 2</u>
The estimate of the root is

$$R_3 = R_2 - \frac{f(R_2)}{f'(R_2)}$$

$$= 13078 - \frac{-1.2914 \times 10^{-8}}{1.9710 \times 10^{-8}}$$

$$= 13078$$

The absolute relative approximate error is

$$\left| \epsilon_a \right| = \left| \frac{13078 - 13078}{13078} \right| \times 100$$

$$= 0.0050097\%$$

The number of significant digits at least correct is 3.

Advantages and Drawbacks of Newton Raphson Method

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Advantages

- Converges fast (quadratic convergence), if it converges.

- Requires only one guess

Drawbacks

1. Divergence at inflection points
    Selection of the initial guess or an iteration value of the root that is close to the inflection point of the function $f(x)$ may start diverging away from the root in ther Newton-Raphson method.

For example, to find the root of the equation $f(x) = (x-1)^3 + 0.512 = 0$.

The Newton-Raphson method reduces to $x_{i+1} = x_i - \dfrac{(x_i^3 - 1)^3 + 0.512}{3(x_i - 1)^2}$.

Table 1 shows the iterated values of the root of the equation.

The root starts to diverge at Iteration 6 because the previous estimate of 0.92589 is close to the inflection point of $x = 1$.

Eventually after 12 more iterations the root converges to the exact value of $x = 0.2$.

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Drawbacks – Inflection Points

Table 1 Divergence near inflection point.

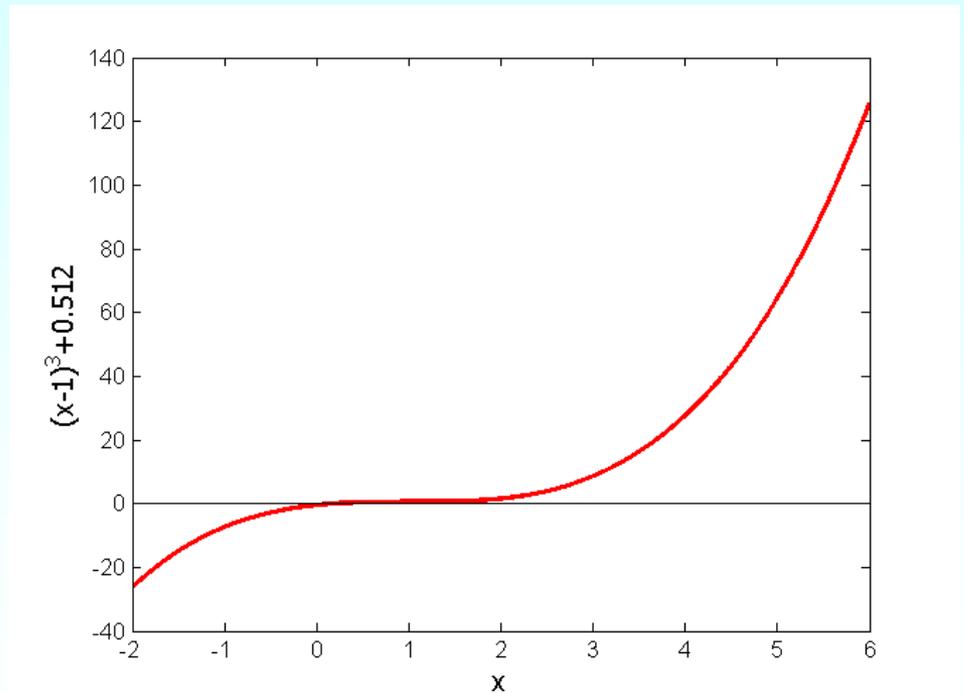| Iteration Number	x_i
0	5.0000
1	3.6560
2	2.7465
3	2.1084
4	1.6000
5	0.92589
6	−30.119
7	−19.746
18	0.2000

Figure 8 Divergence at inflection point for
$$f(x) = (x-1)^3 + 0.512 = 0$$

Drawbacks – Division by Zero

2. Division by zero
   For the equation

$$f(x) = x^3 - 0.03x^2 + 2.4 \times 10^{-6} = 0$$

the Newton-Raphson method reduces to

$$x_{i+1} = x_i - \frac{x_i^3 - 0.03x_i^2 + 2.4 \times 10^{-6}}{3x_i^2 - 0.06x_i}$$

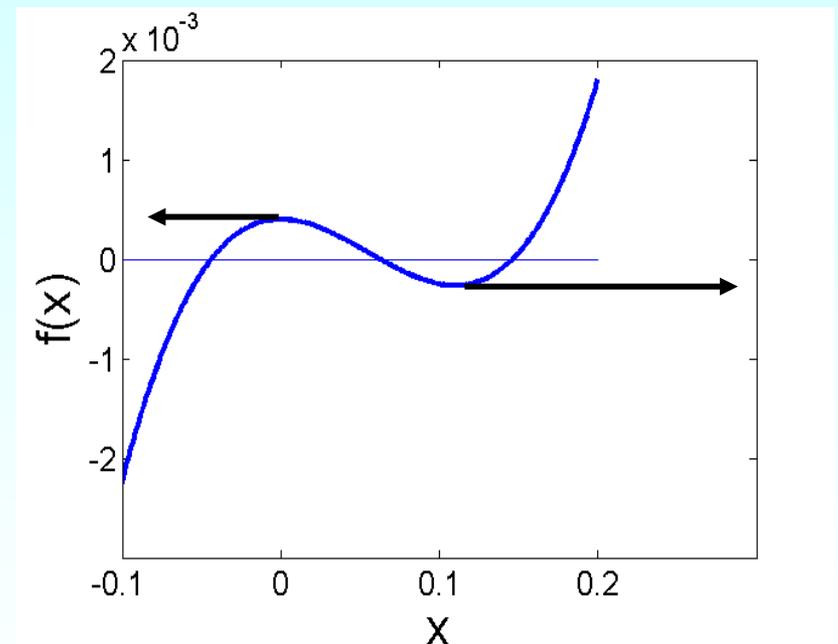For $x_0 = 0$ or $x_0 = 0.02$, the denominator will equal zero.

Figure 9 Pitfall of division by zero or near a zero number

Drawbacks – Oscillations near local maximum and minimum

3. <u>Oscillations near local maximum and minimum</u>

    Results obtained from the Newton-Raphson method may oscillate about the local maximum or minimum without converging on a root but converging on the local maximum or minimum.

    Eventually, it may lead to division by a number close to zero and may diverge.

    For example for $f(x) = x^2 + 2 = 0$ the equation has no real roots.

Drawbacks – Oscillations near local maximum and minimum

Table 3 Oscillations near local maxima and mimima in Newton-Raphson method.

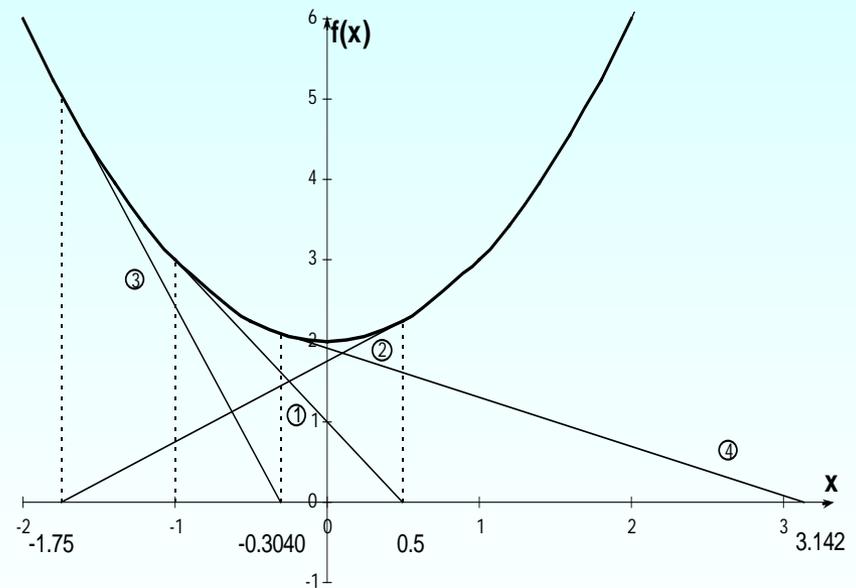| Iteration Number	x_i	$f(x_i)$	$\left|\in_a\right|\%$
0	−1.0000	3.00	
1	0.5	2.25	300.00
2	−1.75	5.063	128.571
3	−0.30357	2.092	476.47
4	3.1423	11.874	109.66
5	1.2529	3.570	150.80
6	−0.17166	2.029	829.88
7	5.7395	34.942	102.99
8	2.6955	9.266	112.93
9	0.97678	2.954	175.96

Figure 10 Oscillations around local minima for $f(x)=x^2+2$.

Drawbacks – Root Jumping

4. <u>Root Jumping</u>
In some cases where the function $f(x)$ is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root.

For example
$$f(x) = \sin x = 0$$

Choose
$$x_0 = 2.4\pi = 7.539822$$

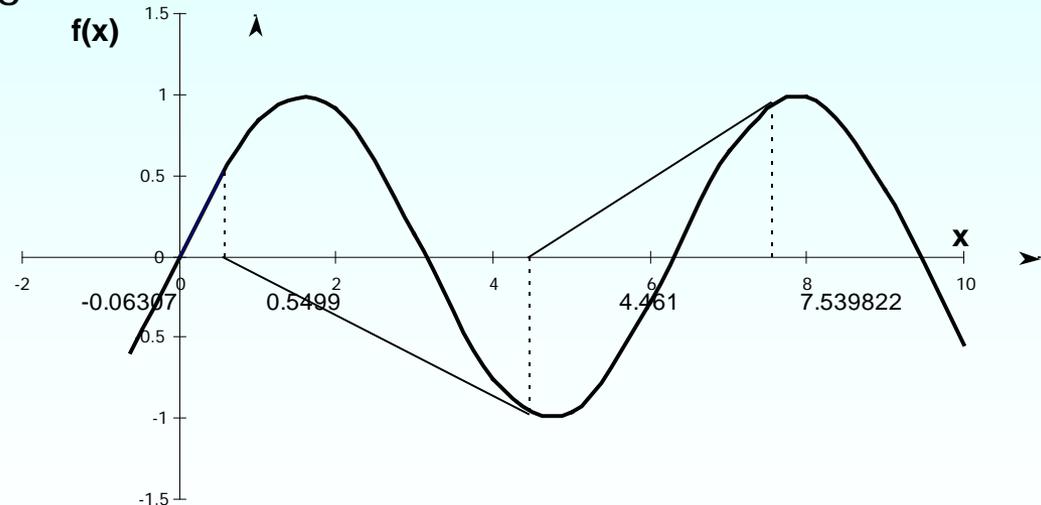It will converge to $x = 0$

instead of $x = 2\pi = 6.2831853$

Figure 11 Root jumping from intended location of root for $f(x) = \sin x = 0$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/newton_raphson.html

THE END